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"Expanded Version of
The Normalized Planck's Probability Distribution, its Derivation, Statistical Properties and Applications in Modeling the Earth's Thermal Atmospheric Emission"

by

M. Shakil[1], E. Villamor[2], T. Hussain[3], M. Ahsanullah[4], and B. M. G. Kibria[5]

[1]Miami Dade College, Hialeah, FL 33012, USA, mshakil@mdc.edu

[2]Department of Mathematics and Statistics / Institute of Environment, Florida International University, Miami, FL 33199, USA, E-mail: villamor@fiu.edu

[3] Department of Statistics, Mirpur University of Science and Technology (MUST), Mirpur 10250, Pakistan, E-mail: tassaddaq.stat@must.edu.pk

[4]Professor Emeritus, Rider University, Lawrenceville, NJ 08648, USA, E-mail: ahsan@rider.edu

[5]Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA, E-mail: kibriag@fiu.edu

Institute of Environment, Florida International University

Miami, FL 33199, USA

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DEDICATION

The authors dedicate this technical note to their families.

Foreword

Planck's Radiation Law is one of the most celebrated achievements in modern physics and one of the cornerstones upon which quantum theory was built. Developed by Max Planck in 1900, the law provides a precise mathematical description of the spectral distribution of electromagnetic radiation emitted by an ideal black body in thermal equilibrium at a given absolute temperature. The successful explanation of black-body radiation not only resolved a major inconsistency in classical physics but also introduced the revolutionary concept of energy quantization, thereby laying the foundation for quantum mechanics. Prior to Planck's work, Kirchhoff established the universality of black-body radiation, while Stefan and Boltzmann developed the relationship between total emitted energy and temperature. Wien obtained the displacement law describing the shift of the spectral peak with temperature. However, classical theories ultimately led to the Rayleigh–Jeans law, which predicted an infinite energy density at high frequencies, a contradiction known as the ultraviolet catastrophe. Planck overcame this difficulty by introducing the assumption that electromagnetic energy is exchanged in discrete packets or quanta. This quantization hypothesis transformed theoretical physics and initiated the development of modern quantum theory. Einstein subsequently extended the concept through the photon theory of light, while Bose and Einstein established its statistical foundation through Bose–Einstein statistics. Beyond its physical significance, Planck's radiation law possesses remarkable mathematical properties. The law exhibits a non-negative continuous functional form, is integrable over its domain, and generates several important results in mathematical physics through differentiation and integration.

In recent years, considerable attention has been devoted to the development of probability distributions generated from physical laws, differential equations, and stochastic models. Many classical probability distributions arise naturally as solutions of differential equations or as normalized versions of physically meaningful functions. The exponential, gamma, Weibull, Rayleigh, Maxwell, and generalized gamma distributions provide notable examples of this connection between mathematical physics and probability theory. The Planck radiation function exhibits several characteristics commonly associated with probability density functions, including positivity, continuity, finite moments, and well-defined shape characteristics. These features suggest that, after appropriate normalization, Planck's radiation law may serve as the basis for constructing a continuous probability distribution. Moreover, the differential structure of Planck's function offers opportunities for deriving probability models through ordinary differential equation techniques. Motivated by these considerations, in the present technical note, we develop a first-order linear differential equation approach for Planck's radiation law and derives an associated continuous probability distribution. Various statistical properties including moments, generating functions, reliability measures, entropy measures, and characterization results are investigated. The proposed framework contributes to the growing literature on distribution generation through differential equations and demonstrates how a fundamental law of quantum physics can be interpreted within a probabilistic setting.

The organization of the chapters is as follows: Section 2 discusses the principal mathematical forms of Planck's radiation law. Section 3 reviews important historical derivations and quantum-statistical developments. Section 4 examines classical limiting laws and associated distributions. In Section 5, we present generating the normalized Planck's continuous probability distribution associated with Planck's radiation law through a first-order linear differential equation approach. Section 6 discusses statistical properties. Section 7 introduces characterization results via truncated moments and reverse hazard rates. In Section 8, we present applications and parameter estimation of the normalized Planck's distribution. Finally, in Section 9, we present concluding remarks. For the sake of completeness, in the Appendix, we have provided the notations, symbols, functions, definitions and mathematical results that are useful in the study of the normalized Planck's probability distribution and applications.

We hope that the findings of this technical note will be useful for the practitioners in various fields of studies and further enhancement of research in the fields of environmental science, quantum physics, statistical mechanics, probability, statistics, and their applications.

The Normalized Planck's Probability Distribution, its Derivation, Statistical Properties and Applications in Modeling the Earth's Thermal Atmospheric Emission

Abstract

Planck's radiation law, one of the most profound achievements in modern physics, describes the spectral distribution of blackbody radiation and lays the foundation for quantum theory. Beyond its fundamental physical significance, it possesses rich mathematical and probabilistic structures. In this paper, we present a comprehensive analysis of the normalized Planck probability distribution associated with Planck's radiation law. A first-order differential equation approach is developed to generate its probability density function. Its statistical properties, including moments, generating functions, entropy measures, and other fundamental distributional characteristics, are derived and analyzed. Furthermore, new characterization results are established using truncated moments and reverse hazard rate functions. The applicability of the proposed framework is demonstrated by modeling the Earth's thermal atmospheric emission using the exact continuous probability density function of the emitted radiation frequency, ν, for a blackbody at physical temperature T. The results highlight the mathematical elegance, probabilistic richness, and practical utility of the normalized Planck probability distribution, providing a rigorous foundation for future research and applications in statistical physics, probability theory, applied mathematics, and related scientific disciplines.

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Keywords: Planck's Radiation Law; Black-Body Radiation; Continuous Probability Distribution; First-Order Linear Differential Equation; Truncated Moments; Reverse Hazard Rate Function.

1. INTRODUCTION

Planck's Radiation Law is one of the most celebrated achievements in modern physics and one of the cornerstones upon which quantum theory was built. Developed by Max Planck in 1900, the law provides a precise mathematical description of the spectral distribution of electromagnetic radiation emitted by an ideal black body in thermal equilibrium at a given absolute temperature. The successful explanation of black-body radiation not only resolved a major inconsistency in classical physics but also introduced the revolutionary concept of energy quantization, thereby laying the foundation for quantum mechanics (Planck [1, 2,1900]; Planck [3, 1901]).

A black body is an idealized physical system that absorbs all incident electromagnetic radiation irrespective of frequency, wavelength, angle of incidence, direction, or polarization and re-emits radiation solely according to its temperature. The distribution of radiant energy emitted by a black body can be expressed either as a function of $frequency\ \nu\ or\ wavelength\ \lambda$, called the Planck's radiation law, respectively given by

$$B(\nu,T) \; = \; (\frac{2h\nu^3}{c^2}) \left[\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right], \tag{1.1}$$

and

$$B(\lambda,T) = \left(\frac{2hc^2}{\lambda^5}\right) \left[\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}\right], \tag{1.2}$$

where h denotes Planck's constant, c represents the speed of light in vacuum, k denotes Boltzmann's constant, and T the black body temperature. These expressions accurately describe the spectral energy distribution of thermal radiation over the entire electromagnetic spectrum and have been verified experimentally with remarkable precision.

Prior to Planck's work, Kirchhoff (Kirchhoff [4, 1860]) established the universality of black-body radiation, while Stefan and Boltzmann developed the relationship between total emitted energy and temperature, known as the Stefan-Boltzmann law (or the T^4 radiation law) which states that the total energy radiated by a surface is directly proportional to the fourth power of its absolute temperature T (Stefan [5, 1879]; Boltzmann [6, 1884]). Wien proposed an empirical law valid at high frequencies to describe the distribution of energy in blackbody radiation (Wien [7, 1897]). However, classical theories ultimately led to the Rayleigh–Jeans law, which predicted an infinite energy density at high frequencies, a contradiction known as the ultraviolet catastrophe (Rayleigh [8, 1900]; Jeans [9, 1905]). To resolve the ultraviolet catastrophe associated with classical radiation theory, Max Planck overcame this difficulty by introducing the assumption that electromagnetic energy is exchanged in discrete packets or quanta, and introduced the radiation law, known as Planck's radiation law (Planck [3, 1901]). It describes the spectral distribution of electromagnetic radiation emitted by a black body in thermal equilibrium at absolute temperature T. By introducing the revolutionary hypothesis that electromagnetic energy is emitted in discrete quanta of magnitude $h\nu$, where h denotes Planck's constant and ν frequency, Planck successfully derived a radiation formula that accurately matched experimental observations across the entire spectrum (Planck [3,1901]), as described below:

$$E = h\nu, \tag{1.3}$$

and

$$\langle E \rangle = \frac{h\nu}{\left(e^{\frac{h\nu}{kT}} - 1\right)}. \tag{1.4}$$

This quantization hypothesis transformed theoretical physics and initiated the development of modern quantum theory. Einstein subsequently extended the concept through the photon theory of light, while Bose and Einstein established its statistical foundation through Bose–Einstein statistics (Einstein [10, 1905]; Bose [11, 1924]). Beyond its physical significance, Planck's radiation law possesses remarkable mathematical properties. The law exhibits a non-negative continuous functional form, is integrable over its domain, and generates several important results in mathematical physics through differentiation and integration. For details on spectral energy density expressed in terms of different spectral variables, we refer to Planck [12, 1914], Saha and Srivastava [13, 1958], Robitaille [14, 2008; 15, 2009], Crepeau [16, 2009], Kramm and Mölders [17, 2009], Marr and Wilkin [18, 2012], Kafri [19, 2019], Walker [20, 2022], Mavani and Singh [21, 2024], Küçük [22, 2025], Samanta [23, 2025], Dash and Panigrah [24, 2025], Gomez-Uribe [25, 2026], and references therein.

In recent years, considerable attention has been devoted to the development of probability distributions generated from physical laws, differential equations, and stochastic models. Many classical probability distributions arise naturally as solutions of differential equations or as normalized versions of physically meaningful functions. The exponential, gamma, Weibull, Rayleigh, Maxwell, and generalized gamma distributions provide notable examples of this connection between mathematical physics and probability theory. The Planck radiation function exhibits several characteristics commonly associated with probability density functions, including positivity, continuity, finite moments, and well-defined shape characteristics. These features suggest that, after appropriate normalization, Planck's radiation law may serve as the basis for constructing a continuous probability distribution. Moreover, the differential structure of Planck's function offers opportunities for deriving probability models through ordinary differential equation techniques. Motivated by these considerations, the present article develops a first-order linear differential equation approach for Planck's radiation law and derives an associated continuous probability distribution. Various statistical properties including moments, generating functions, reliability measures, entropy measures, and characterization results are investigated. The proposed framework contributes to the growing literature on distribution generation through differential equations and demonstrates how a fundamental law of quantum physics can be interpreted within a probabilistic setting. The remainder of the paper is organized as follows. Section 2 discusses the principal mathematical forms of Planck's radiation law. Section 3 reviews important historical derivations and quantum-statistical developments. Section 4 examines classical limiting laws and associated distributions. In Section 5, we present generating the normalized Planck's continuous probability distribution associated with Planck's radiation law through a first-order linear differential equation approach. Section 6 discusses statistical properties. Section 7 introduces characterization results via truncated moments and reverse hazard rates. In Section 8, we present applications and parameter estimation of the normalized Planck's distribution. Finally, in Section 9, we present concluding remarks.

2. MATHEMATICAL FORMS OF PLANCK'S RADIATION LAW

Planck's law admits several mathematically equivalent formulations depending on whether the radiation is expressed in terms of frequency ν or wavelength λ and whether one considers energy density or spectral radiance.

2.1 Spectral Energy Density in Terms of Frequency

The spectral energy density per unit frequency interval ν is

$$u(\nu, T) = \left(\frac{8\pi h\nu^3}{c^3}\right)\left(\frac{1}{e^{\left(\frac{h\nu}{k_B T}\right)} - 1}\right), \quad \nu > 0.$$

2.2 Spectral Radiance in Terms of Frequency

The spectral radiance form is

$$B(\nu, T) = \left(\frac{2\pi h\nu^3}{c^2}\right)\left(\frac{1}{e^{\left(\frac{h\nu}{k_B T}\right)} - 1}\right).$$

2.3 Spectral Energy Density in Terms of Wavelength

To transform Planck's Law from frequency (ν) to wavelength (λ), since we are interested in the size of the interval irrespective of its direction (whether frequency increases or decreases), we apply the change of variables $\nu = c/\lambda$ and take the absolute value of the derivative, that is, $\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2}$, also known as the Jacobian. Then, the wavelength form of Planck's Law becomes

$$u(\lambda, T) = u(\nu, T) \cdot \left|\frac{d\nu}{d\lambda}\right| = \left(\frac{8\pi h\nu^3}{c^3}\right)\left(\frac{1}{e^{\left(\frac{h\nu}{k_B T}\right)} - 1}\right) \cdot \frac{c}{\lambda^2}$$

$$= \left(\frac{8\pi hc}{\lambda^5}\right)\left(\frac{1}{e^{\left(\frac{hc}{\lambda k_B T}\right)} - 1}\right), \quad \lambda > 0.$$

2.4 Spectral Radiance in Terms of Wavelength

Similarly,

$$B(\nu, T) = \left(\frac{2hc^2}{\lambda^5}\right)\left(\frac{1}{e^{\left(\frac{hc}{\lambda k_B T}\right)} - 1}\right).$$

These representations are not identical under simple substitution because the differential elements $d\nu$ and $d\lambda$ are related nonlinearly. Consequently, maxima of the frequency and wavelength forms occur at different points.

The Planck distribution possesses the following important qualitative properties:

(a) positivity,

(b) continuity on $(0, \infty)$,

(c) exponential decay at high frequencies,

(d) integrability over the entire spectrum, and

(e) temperature dependence through scaling transformations.

These properties ensure that the normalized Planck function defines a proper continuous probability density function.

3. HISTORICAL DEVELOPMENT AND DERIVATIONS

The development of Planck's radiation law marks one of the most important turning points in modern physics and the origin of quantum theory. In what follows, we present a historical development of Planck's radiation law.

3.1 Classical Background

Prior to Planck's work, classical electrodynamics and statistical mechanics attempted to describe black-body radiation using equipartition arguments. The Rayleigh–Jeans law successfully described long wavelengths but diverged catastrophically at short wavelengths, leading to the ultraviolet catastrophe.

3.2 Max Planck's Quantization Hypothesis

In 1900, Max Planck proposed that electromagnetic oscillators could emit or absorb energy only in discrete packets (Planck [1, 2,1900]):

$$E = nh\nu, n = 0,1,2, \dots .$$

Using combinatorial entropy arguments and Boltzmann's relation between entropy and probability, Planck obtained the correct radiation spectrum (Planck [3, 1901]).

3.3 Bose's Statistical Derivation (1924)

Satyendra Nath Bose (Bose [26, 1924]) reformulated black-body radiation using quantum statistics. Bose treated photons as indistinguishable particles and partitioned phase space into elementary cells of volume h^3. By counting the number of possible photon configurations and maximizing entropy, Bose derived Planck's law without reference to classical electrodynamics. Einstein immediately recognized the importance of Bose's method and generalized it to material particles, leading to Bose–Einstein statistics. For details, the interested readers are also referred to Mavani and Singh [21, 2024], Küçük [22, 2025], Samanta [23, 2025], Dash and Panigrah [24, 2025], Gomez-Uribe [25, 2026], and references therein.

3.4 Jordan's Quantum Field Formulation

Pascual Jordan (Jordan [27, 1927]) further extended the theoretical foundations by quantizing electromagnetic field modes using matrix mechanics and second quantization. Jordan demonstrated that field fluctuations and wave-particle duality naturally produced Planck's spectrum.

3.5 Later Statistical Developments

The legacy of Planck, Einstein, and Bose paved the way for modern quantum electrodynamics. Subsequently, researchers like N. S. N. Nath (Nath [28, 1936]) and others in the mid-to-late 20th century further formalized quantum statistics, light scattering (like the Raman effect), and many-body interactions to explain the fundamental behaviors of radiation and quantum gases and investigated generalized quantum distributions and their applications to systems obeying exchange symmetries. These studies confirmed the universal role of Bose–Einstein statistics in radiation phenomena and condensed matter systems.

4. DERIVATIONS OF CLASSICAL LAWS AND ASSOCIATED DISTRIBUTIONS

Several important physical laws arise as limiting forms or modifications of Planck's radiation law. In this section, we discuss classical limiting laws and associated distributions.

4.1 Rayleigh–Jeans Law

The Rayleigh–Jeans law approximates blackbody radiation at low frequencies (Rayleigh [8, 1900]; Jeans [9, 1905]). It is derived from Planck's law by expanding the exponential term for the condition $h\nu << k_B T$. As stated above (Sub-Section 2.1), since Planck's spectral energy density for blackbody radiation relates the energy density $u(\nu, T)$ to the frequency ν and absolute temperature T:

$$u(\nu,T) = \left(\frac{8\pi h\nu^3}{c^3}\right)\left(\frac{1}{e^{\left(\frac{h\nu}{k_BT}\right)}-1}\right), \quad \nu > 0,$$

where h is the Planck's constant, k_B is the Boltzmann's constant, and c is the speed of light, therefore, at low frequencies (where $h\nu << k_BT$), the exponent $\frac{h\nu}{k_BT}$ is very small. Hence, using the Taylor series expansion for the exponential function, we get:

$$e^{\left(\frac{h\nu}{k_BT}\right)} \approx 1 + \frac{h\nu}{k_BT}.$$

Substituting this into Planck's formula and simplifying yields

$$u(\nu,T) \approx \left(\frac{8\pi\nu^2}{c^3}\right) k_BT,$$

which is the Rayleigh–Jeans law.

Similarly, as stated in Sub-Section 2.3, the Rayleigh–Jeans law can be expressed in terms of wavelength λ as follows:

$$u(\lambda,T) \approx \frac{8\pi k_BT}{\lambda^4}.$$

4.2 Wien Approximation

For high frequencies satisfying $h\nu >> k_BT$, the term "−1" in the denominator of Planck's formula for frequency ν becomes negligible. Hence,

$$u(\nu,T) \approx \left(\frac{8\pi h\nu^3}{c^3}\right) e^{\left(-\frac{h\nu}{k_BT}\right)},$$

which is Wien's radiation law (Wien [7, 1897]).

4.3 Bose–Einstein, Maxwell–Boltzmann and Fermi–Dirac Distributions

In what follows, we will discuss Bose–Einstein, Maxwell–Boltzmann and Fermi–Dirac Distributions as limiting forms or modifications of Planck's radiation law. The applications of Maxwell–Boltzmann distribution can be found in nuclear magnetic resonance spectroscopy. As pointed out by Oldham *et al.* [29, 2009], the Maxwell–Boltzmann distribution describes the fall off in atmospheric pressure with height, as well as many other phenomena in which a force is opposed by thermal agitation. The speeds of gas molecules in two and three-dimensional kinetic theory of gases also follow the Maxwell–Boltzmann distribution. On the other hand, the statistical behaviors of subatomic particles, such as fermions (particles with half-integer spin) and bosons (integer spin particles), can be described by the Fermi-Dirac and Bose-Einstein distributions respectively, see, for example, Kittel [30, 2004], Pathria and Beale [31, 2011], Shepherd [32, 2013], and Ahsanullah and Shakil [33, 2015], among others.

4.3.1 Bose–Einstein Distribution

Indian physicist Satyendra Nath Bose derived Planck's law using a new statistical approach for indistinguishable particles with integer spin (later named bosons). Albert Einstein expanded this to matter, predicting the Bose-Einstein Condensate (BEC) - a state of matter where particles cooled to near absolute zero merge into a single quantum state (Einstein [10, 1905]; Bose [11, 1924]).

The Bose–Einstein occupancy distribution is given by

$$f(\varepsilon) = \frac{1}{e^{\left(\frac{\varepsilon-\mu}{k_B T}\right)} - 1},$$

where μ is the chemical potential and ε denotes the quantum state of energy (that is, energy per photon). For photons, the chemical potential $\mu = 0$ and $\varepsilon = h\nu$. See Kittel [30, 2004], Pathria and Beale [31, 2011], and Shepherd [32, 2013].

Now, in Planck's law, the density of states per unit volume as a function of frequency ν represents the number of electromagnetic field modes (or "spaces" available for photons) per unit volume in a frequency interval and for a $3D$ cavity, it is given by:

$g(\nu) = \frac{8\pi\nu^2}{c^3}$, where ν is frequency and c is the speed of light.

Multiplying the above three expressions, density of states $g(\nu)$, energy per photon ε, and Bose-Einstein distribution $f(\varepsilon)$, together, yields the formula for Planck's spectral energy density in terms of frequency ν.

4.3.2 Maxwell–Boltzmann Distribution

In the dilute classical limit where occupancy numbers are very small,

$$e^{\left(\frac{\varepsilon}{k_B T}\right)} >> 1,$$

and the Bose–Einstein distribution reduces to the Maxwell–Boltzmann form

$f(\varepsilon) \approx e^{\left(-\frac{\varepsilon}{k_B T}\right)}$, see Young *et al.* [34, 2008]; Oldham *et al.* [29, 2009].

4.3.3 Fermi–Dirac Distribution

For fermionic particles obeying the Pauli exclusion principle*,

$$f(\varepsilon) = \frac{1}{e^{\left(\frac{\varepsilon-\mu}{k_B T}\right)}+1}.$$

The " $+1$" denominator distinguishes fermionic statistics from bosonic statistics. See Kittel [30, 2004], Pathria and Beale [31, 2011], Shepherd [32, 2013], and Young *et al.* [34, 2008].

***Note:** The Pauli exclusion principle is a fundamental quantum mechanics rule stating that no two identical particles with half-integer spins (called fermions, such as electrons) can occupy the same exact quantum state simultaneously. In an atom, this means no two electrons can share the same set of four quantum numbers.

5. A FIRST ORDER DIFFERENTIAL EQUATION APPROACH TO GENERATE THE EXACT PHYSICAL PLANCK'S CONTINUOUS PROBABILITY DENSITY FUNCTION

Probability distributions play a central role in statistical analysis and mathematical modeling, enabling researchers to describe uncertainty, variability, and randomness in various real-world phenomena. Continuous probability distributions are often used when dealing with data that can take any value within a range or interval. Over the years, many methods have been proposed to generate such distributions, with the differential equation approach standing out as one of the most powerful and flexible techniques. Karl Pearson, in his work in the late 19th century, introduced the method of generating probability distributions through first-order linear differential equations, given by

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{-(x+a)}{c_0 + c_1 x + c_2 x^2}, \tag{5.1}$$

where a, c_0, c_1, and c_2 represent real parameters, ensuring that f serves as a probability density function, known as the Pearson system of distributions (Pearson [35, 1895; 36, 1901; 37, 1916]). This approach provides a framework for deriving distributions that satisfy certain conditions or properties, such as moments, skewness, and kurtosis. Since Pearson's pioneering work, the differential equation approach has evolved significantly, and it continues to be a rich area of research in modern statistical theory. Over time, this approach has been extended to more general forms, for example, the generalized Pearson system of distributions could be generated by the following differential equations:

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, \tag{5.2}$$

where $m, n \in N_0$ (set of non-negative integers), and the coefficients a_j and b_j are real parameters. By proper choices of these parameters, a vast majority of continuous probability density functions could be generated from the above generalized Pearson system of distributions. In the context of the generalized Pearson family, the PDF $f(x)$ represents the distribution of a random variable X, (Shakil and Kibria [38, 2010], Shakil *et al*. [39, 2010], Ahsanullah *et al*. [40, 2013], Singh [41, 2018], Singh and Zhang [42, 2020], Kibria *et al*. [43, 2021], Ahsanullah and Shakil [44, 2023], Shakil *et al*. [45, 2024]), and Shakil *et al*. [46, 2025]).

5.1 Derivation of Planck's Normalized Probability Density Function (PDF)

In this research note, without loss of generality, we examine the differential equation approach to generating the exact physical univariate Planck's normalized continuous probability density function of frequency ν. In physics, when referring to Planck's probability density function (PDF), we are generally describing the statistical distribution of energy states across microscopic particles in a system, such as photon gas. The derivation begins with Pearson's differential equation of the form (5.1), a framework frequently applied to probability distributions, by adjusting the parameters c_0, c_1, and c_2 defining the variance and skewness. The unique continuous solution to the above first-order linear differential equation (5.1) derives Planck's distribution.

A normalized Planck's density defines a valid continuous probability distribution over the positive real line. To model a physical system where the derivative is proportional to a function of the random variable X, let $f(x)$ denote a generalized Planck-type density. Now, recalling from (2.1), Planck's law for spectral energy density as a function of frequency ν is expressed as

$$u(\nu) = \left(\frac{8\pi h\nu^3}{c^3}\right)\left(\frac{1}{e^{\left(\frac{h\nu}{k_B T}\right)} - 1}\right), \quad \nu > 0.$$

To turn this into a standard probability distribution $f(x)$ over a dimensionless variable x, let $x = \frac{h\nu}{k_B T}$. Note that, in Planck's distribution as defined above, the independent variable x (defined as $x \ = \frac{h\nu}{k_B T}$) is dimensionless because it represents the ratio of a photon's energy to thermal energy. This ensures mathematical validity when exponentiating (as in e^x) and allows the probability density function $f(x)$ to be universally applicable. The primary reasons for this dimensionless consideration include:

- **The Ratio of Like Quantities:** The independent variable x is defined as $x = \frac{h\nu}{k_B T}$ or $x = \frac{hc}{\lambda \, k_B \, T}$. The numerator (energy of a photon) and the denominator (thermal energy of the system) both carry units of Joules. Because they are divided, the units cancel out, making a pure number.
- **Mathematical Constraints on Functions:** Also, note that we cannot mathematically evaluate functions like exponentials (e.g., e^x), logarithms (lnx), or trigonometric functions on quantities with physical dimensions. This is because these functions are defined by their Taylor series expansions (e.g., $e^x = 1 + \frac{x^2}{2!} + \cdots$). Adding a dimensionless number (like 1) to a quantity with dimensions (like meters or Joules) is physically meaningless. Therefore, the argument must be dimensionless.
- **Normalization of Probability:** When using the Planck function as a Probability Density Function (PDF), it must satisfy the normalization condition $\int_0^\infty f(x)\, dx = 1$. Because the differential dx is dimensionless, $f(x)$ becomes a dimensionless density (probability per unit interval of x), allowing physicists to compute probabilities independently of the specific temperature or frequency range.

Now, suppose $f(x)$ satisfies the following first-order linear differential equation

$$\frac{df(x)}{dx} + \ [a(x) + b(x)]f(x) = 0, \qquad (5.3)$$

where $a(x)$ and $b(x)$ are continuous coefficient functions determined by the spectral structure of the density. For the Planck's density, by suitably choosing $a(x)$ and $b(x)$ in (5.3), and the parameters c_0, c_1, and c_2 defining the variance and skewness as provided in (5.1), let

$$a(x) = \frac{3}{x}, \text{ and } b(x) = \frac{e^{x}}{(e^{x}-1)},$$

where, x corresponds to the energy of a photon or particle, and embodies the thermal energy scale $k_B T$, given by $x = \frac{h\nu}{k_B T}$. Then, substituting these for $a(x)$ and $b(x)$ in (5.3), we obtain a first-order linear differential equation given by

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{3}{x} - \frac{e^{x}}{(e^{x}-1)},$$

or,

$$\frac{dln(f(x))}{dx} = \frac{3}{x} - \frac{e^{x}}{(e^{x}-1)}. \tag{5.4}$$

The equation (5.4), unlike the standard Pearson system, contains exponential functions to correctly model quantum discrete energy constraints. Integrating both sides of (5.4) with respect to x, using the u-substitution $u = e^{x} - 1 \Rightarrow du = e^{x}dx$, and simplifying, we obtain

$$ln(f(x)) = ln\left(C\frac{x^3}{e^{x}-1}\right), \tag{5.5}$$

where C is the constant of integration and is to be determined.

Now, from (5.5), after taking the exponential of both sides to eliminate the natural logarithms, we obtain

$$f(x) = C\frac{x^3}{e^x-1}. \tag{5.6}$$

To find the value of C, the total area under the probability density function (PDF) must equal 1 over its entire domain ($x > 0$). Thus, from (5.6), we have

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty C\frac{x^3}{e^x-1}dx = 1 \Rightarrow C\int_0^\infty \frac{x^3}{e^x-1}dx = 1. \tag{5.7}$$

Now to make the integrand in (5.7) integrable, we first expand the expression $\frac{1}{e^x-1}$ as a geometric series as follows:

$$\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^{\infty} e^{-\mathrm{n}x}. \tag{5.8}$$

Placing the series (5.8) back for the integral in (5.7) and swap the order of integration and summation, we obtain

$$C\sum_{n=1}^{\infty}\int_0^\infty x^3 e^{-nx}dx = 1. \tag{5.9}$$

To evaluate the integral in (5.9), let $u = nx \Rightarrow dx = \frac{du}{n}$, so that

$$\int_0^\infty x^3 e^{-nx} dx = \frac{1}{n^4}\int_0^\infty u^3 e^{-u} du = \frac{1}{n^4}\Gamma(4) = \frac{1}{n^4}(3!) = \frac{6}{n^4}. \quad (5.10)$$

Then, using the definition of the standard gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \mathcal{R}(z) > 0,$$

and, substituting (5.10) in (5.9), we obtain

$$C \sum_{n=1}^\infty \frac{6}{n^4} = 1, \text{ or } 6C \sum_{n=1}^\infty \frac{1}{n^4} = 1,$$

from which, on using the definition of the standard Riemann Zeta function $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, $\mathcal{R}(s) > 1$, (Gradshteyn and Ryzhik [46, 2007]**)**, we obtain

$$6C\zeta(4) = 1 \Rightarrow C = \frac{1}{6\,\zeta(4)}.$$

Since the known exact value for $\zeta(4)$ is $\frac{\pi^4}{90}$, hence using this value in the above expression for the constant C and simplifying, we have

$$C = \frac{15}{\pi^4}.$$

Thus, substituting this value of C into our general solution (5.6) gives the normalized dimensionless PDF, as follows:

$$f(x) = \frac{15}{\pi^4}\frac{x^3}{e^x - 1}, \; x > 0. \quad (5.11)$$

Now, to transform back (5.11) to physical variables, we recall from the above that the dimensionless variable was originally defined to scale photon frequency against the thermal energy environment, $x = \frac{h\nu}{k_B T}$. Hence, to express this as a physical probability density function with respect to frequency, that is, $g(\nu)$, we must apply a change of variables conservation step as follows:

$$g(\nu)d\nu = f(x)dx \Rightarrow g(\nu) = f(x)\left|\frac{dx}{d\nu}\right|. \quad (5.12)$$

From (5.12), on calculating the derivative $\left|\frac{dx}{d\nu}\right| = \left|\frac{h}{k_B T}\right|$ and substituting back the physical variables, $x = \frac{h\nu}{k_B T}$, after simplification, we easily obtain

$$g(\nu) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}} - 1}, \; \nu > 0. \quad (5.13)$$

Now we shall show that the integral (5.13) of the normalized probability density function for Planck's law of radiation in terms of frequency ν evaluates to exactly 1, that is, we shall show that

$$\int_0^\infty g(\nu)d\nu = \int_0^\infty \frac{15}{\pi^4}\left(\frac{h}{k_BT}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_BT}}-1}d\nu = 1, \tag{5.14}$$

which represents the total probability of all frequencies for blackbody radiation.

For this, let $x = \frac{h\nu}{k_BT} \Rightarrow \nu = \frac{k_BT}{h}x \Rightarrow d\nu = \frac{k_BT}{h}dx$. Now substituting for x and $d\nu$ into the original left side integral of (5.14) and factoring out the constants, we obtain

$$\begin{aligned}\int_0^\infty g(\nu)d\nu &= \int_0^\infty \frac{15}{\pi^4}\left(\frac{h}{k_BT}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_BT}}-1}d\nu \\ &= \frac{15}{\pi^4}\left(\frac{h}{k_BT}\right)^4\left(\frac{k_BT}{h}\right)^4\int_0^\infty \frac{x^3}{e^x-1}dx \\ &= \frac{15}{\pi^4}\int_0^\infty \frac{x^3}{e^x-1}dx, \end{aligned} \tag{5.15}$$

which, on using the well-known the Bose-Einstein integral, defined in terms of the Gamma function (Γ) and the Riemann Zeta function (ζ) by

$$\int_0^\infty \frac{z^{s-1}}{e^z-1}dz = \Gamma(s)\,\zeta(s),\ \mathcal{R}(s) > 1, \tag{5.16}$$

where $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ is the Riemann Zeta function and $\Gamma(s) = \int_0^\infty z^{s-1}e^{-z}dz$, is the gamma function (Gradshteyn and Ryzhik [47, 2007]**)**, we obtain

$$\int_0^\infty g(\nu)d\nu = \frac{15}{\pi^4}\Gamma(4)\,\zeta(4), \tag{5.17}$$

where the value of the gamma function is $\Gamma(4) = 3! = 6$, and $\zeta(4) = \frac{\pi^4}{90}$ is a known value of the Riemann Zeta Function. Substituting these values in (5/17), it simplifies to

$$\int_0^\infty g(\nu)d\nu = 1,$$

fulfilling all properties of a continuous probability density function. That is, the density function $g(\nu)$ satisfies:

(i) $g(\nu) \geq 0$,

(ii) $\int_0^\infty g(\nu)\, d\nu = 1$.

Hence, the resulting function, $g(\nu)$, in (5.13) defines the required exact physical Planck's continuous probability density function of frequency ν. Obviously, the probability density function (PDF), $g(\nu)$, in (5.13), represents the relative likelihood of a blackbody emitting radiant energy at a specific frequency for a given temperature, where the total area under the asymmetric curve always sums to exactly 100% of the emitted energy. Moreover, the normalized Planck's distribution represents the probability density associated with photons occupying specific frequency intervals. It therefore plays a role analogous to the Maxwell–Boltzmann speed distribution for molecular gases.

5.2. Derivation of Planck's Cumulative Distribution Function (CDF)

In what follows, we derive the cumulative distribution function (CDF) $G(\nu)$ of Planck's normalized blackbody radiation law by frequency ν, given by

$$G(\nu) = \int_0^\nu g(\nu) d\nu = \int_0^\nu \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1} d\nu. \tag{5.14}$$

Letting $x = \frac{h\nu}{k_B T} \Rightarrow \nu = \frac{k_B T}{h} x \Rightarrow d\nu = \frac{k_B T}{h} dx$ in (5.14), and simplifying, we obtain

$$G(\nu) = \frac{15}{\pi^4} \int_0^{\frac{h\nu}{k_B T}} \frac{x^3}{e^x - 1} dx, \tag{5.15}$$

from which, on using the definition of the nth order Debye function given by

$D_n(z) = \frac{n}{z^n} \int_0^z \frac{t^n}{e^t - 1} dt$, (Gradshteyn and Ryzhik [47, 2007]; Abramowitz and Stegun [48, 1972]), when $n = 3$, after simplification, we obtain

$$G(\nu) = \frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 D_3\left(\frac{h\nu}{k_B T}\right) \nu^3. \tag{5.16}$$

Using the definition of the third-order Debye function $D_3(x)$ given by

$$D_3(z) = \frac{3}{z^3} \int_0^z \frac{t^3}{e^t - 1} dt,$$

in (5.16), simplifying $G(\nu)$, differentiating $G(\nu)$ with respect to ν, and then taking the derivative of our simplified expression using the Fundamental Theorem of Calculus (Leibniz Integral Rule) and the chain rule, it can easily be seen that

$$\frac{dG(\nu)}{d\nu} = g(\nu) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1}, \quad \nu > 0.$$

Hence, the resulting function, $G(\nu)$, in (5.16), defines the required exact physical Planck's cumulative distribution function (CDF) of frequency ν. It is obvious that the above expression for the normalized cumulative distribution function (CDF), $G(\nu)$, in (5.16), defines the fraction of total spectral radiant energy density emitted by a blackbody up to a specific frequency ν at a thermodynamic temperature T.

5.3. Visual display of Planck's PDF and CDF

Using some Python codes to generate the precise visual graph of both the PDF (5.13) and CDF (5.16) curves, we have sketched the following figures, Figure 1 (a and b), which provide a complete visual analysis and behavior of the physical Planck's continuous probability density and cumulative distribution functions, for some selected special values of the frequency ν, across three values of the temperature T, $3000K$, $4000K$, and $5000K$, respectively.

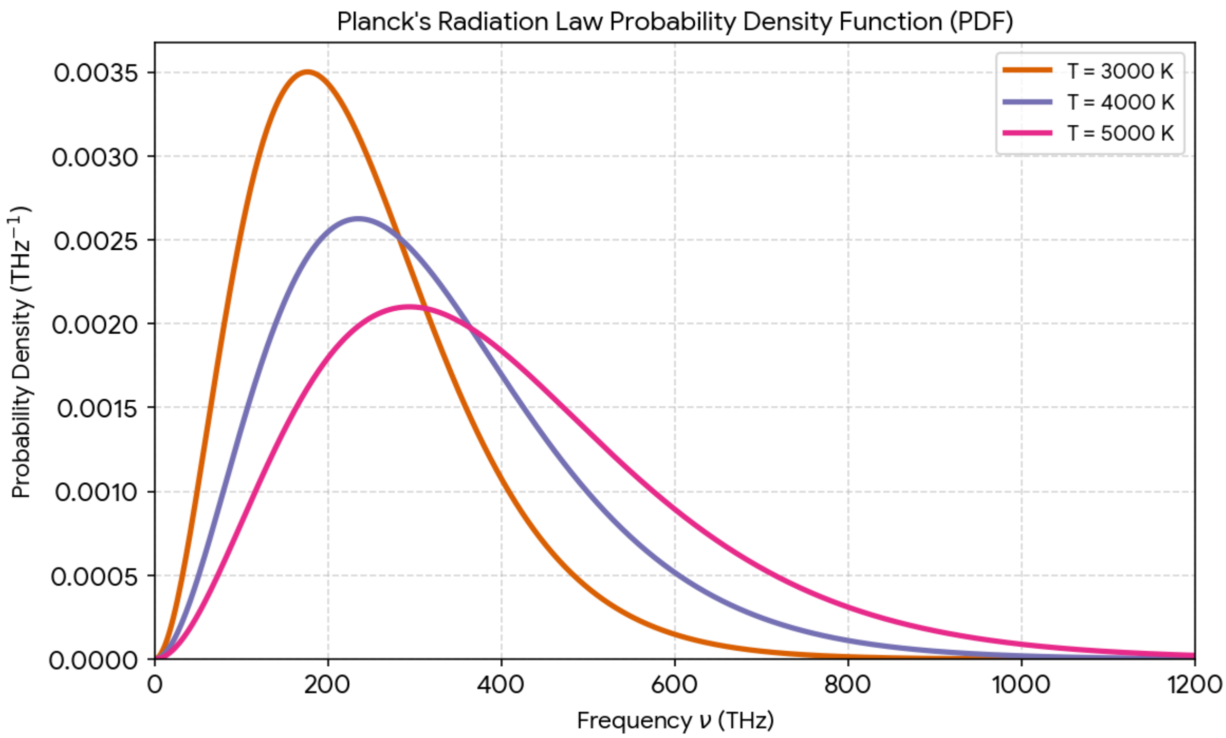

Figure 1 (a): Planck's continuous PDF, $g(\nu)$

Visual Analysis of Planck's continuous PDF, $g(\nu)$: From Figure 1 (a), we observe the following:

- Total Area Equals 1: Because these curves represent a normalized probability distribution, the total area under each individual curve is exactly 1.0, that is, The total area under each individual curve from $\nu = 0$ to $\nu = \infty$ always equals 1.0.
- Asymmetric Profile: The curves rise rapidly at low frequencies ($\sim\nu^3$) and decay exponentially at high frequencies.
- Peak Scaling: The height of the peak increases proportionally with the temperature, that is, as the temperature increases from 3000K to 5000K, the peak frequency moves linearly to the right.

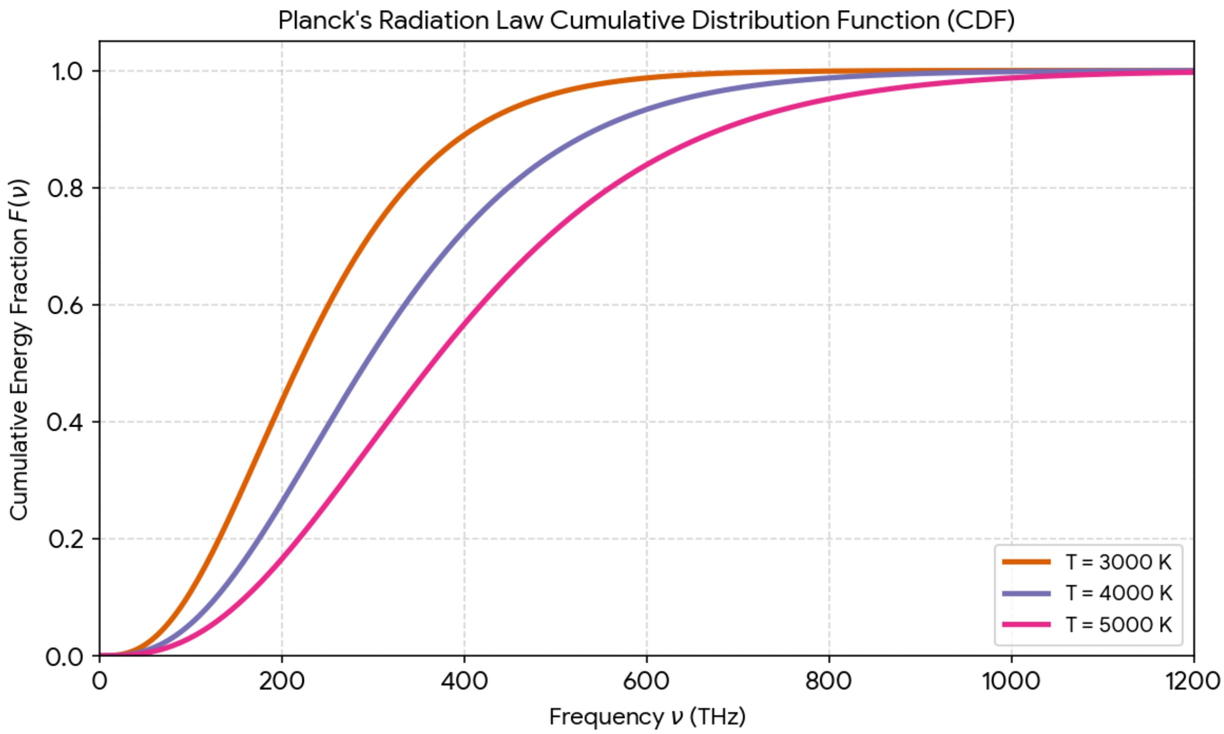

Figure 1 (b): Planck's continuous CDF, $G(\nu)$

Visual Analysis of Planck's continuous CDF, $G(\nu)$: From Figure 1 (b), we observe the following:

- Slope changes: The curves rise sharply as they enter the peak emission spectrum and flatten out asymptotically toward 1.0 as the frequency approaches infinity.
- Energy thresholds: At $3000\ K$, 90% of the energy is reached at around $410\ THz$. At $5000\ K$, that same 90% milestone requires reaching $683\ THz$.

Remark: We can similarly derive exact physical Planck's continuous probability density and cumulative distribution functions for other forms.

5.4. Derivation of Planck's Failure Rate Function (or Hazard Function)

The failure rate function, or hazard function, denoted as $\lambda(x)$, is defined as the ratio of the probability density function of an item's life-length to its reliability function. It represents the probability of an item's failure in a small interval given that it has survived up to that point, or, more specifically, it measures the instantaneous risk that an item, system, or organism fails at a specific time, given that it has survived up to that exact moment. It is a core concept in reliability engineering and survival analysis. The Planck's probability distribution provides a flexible framework for modeling lifetime data encountered in reliability engineering and survival analysis. Its versatility allows it to accommodate a wide range of failure-rate patterns observed in practical applications, such, in environmental science and remote sensing, earth's emitted spectral radiance and the outgoing longwave radiation (OLR) can be modeled using Planck's distribution. Several reliability measures are particularly useful for assessing the lifetime characteristics of the distribution. These include the survival (reliability) function, the hazard (failure) rate function, the cumulative hazard function, defined as $\mathrm{H(x)} = -\ln(\mathrm{S(x)})$, and Mill's ratio, $\mathrm{m(x)} = \frac{\mathrm{S(x)}}{\mathrm{f(x)}}$. Together, these functions describe the probability of survival, the instantaneous risk of failure, the accumulated risk over time, and the relationship between the survival and density functions. Consequently, they play an important role in reliability assessment, time-to-event modeling, and statistical inference. As defined above, for the Planck's probability density function (PDF)

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}}-1}, \quad x > 0,$$

and the cumulative distribution function (CDF)

$$P(x) = \frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^3 D_3\left(\frac{hx}{k_B T}\right),$$

where $h =$ Planck constant, $k_B =$ Boltzmann constant, and $T\ (> 0) =$ absolute temperature, and $D_3(\cdot)$ is the third-order Debye function, the reliability (or survival) function (the probability that the subject is still functioning at time $t = x$) is given by

$$S(x) = 1 - P(x) = 1 - \frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^3 D_3\left(\frac{hx}{k_B T}\right).$$

Hence the failure rate function (or hazard function) is given by

$$\lambda(x) = \frac{p(x)}{1 - P(x)} = \frac{\frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{hx/(k_B T)}-1}}{1 - \frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^3\, D_3\left(\frac{hx}{k_B T}\right)}, \qquad x > 0\,, \tag{5.17}$$

which is the required failure rate function (or hazard function) of Planck's probability density function. Also, we have the cumulative hazard function given by

$$\mathrm{H(x)} = -\ln(\mathrm{S(x)}) = -\ln\left(1 - \frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^3 D_3\left(\frac{hx}{k_B T}\right)\right), \ \ x > 0\,, \tag{5.18}$$

and Mill's ratio is given

$$m(x) = \frac{S(x)}{f(x)} = \frac{1-\frac{5}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^3 D_3\left(\frac{hx}{k_B T}\right)}{\frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}}-1}}, \ x > 0 , \tag{5.19}$$

Visual display of Planck's Hazard Function Plot: Now, without loss of generality, introducing the dimensionless variable $z = \frac{h\,x}{k_B\,T}$, the hazard function $\lambda(x)$ as given above in (5.17) may be expressed as

$$\lambda(x) = \frac{h}{k_B T} \ \frac{\frac{15}{\pi^4}\frac{z^3}{e^z-1}}{1-\frac{5}{\pi^4}z^3 D_3(z)} . \tag{5.20}$$

Obviously, the above form of the hazard function (5.20) is convenient for theoretical analysis because all shape properties depend only on z. Using some Python codes to generate the precise visual graph of Planck's hazard function (5.20) curves, we have plotted the dimensionless hazard function, $\frac{\lambda(x)}{h/(k_B T)}$, as a function of $z = \frac{h\,x}{k_B\,T}$, in the following figure, Figure 2, which provides a complete visual analysis and behavior of the physical Planck's hazard function (5.20) curves, for some selected special values of the frequency $\nu = x$, across some values of the temperature T.

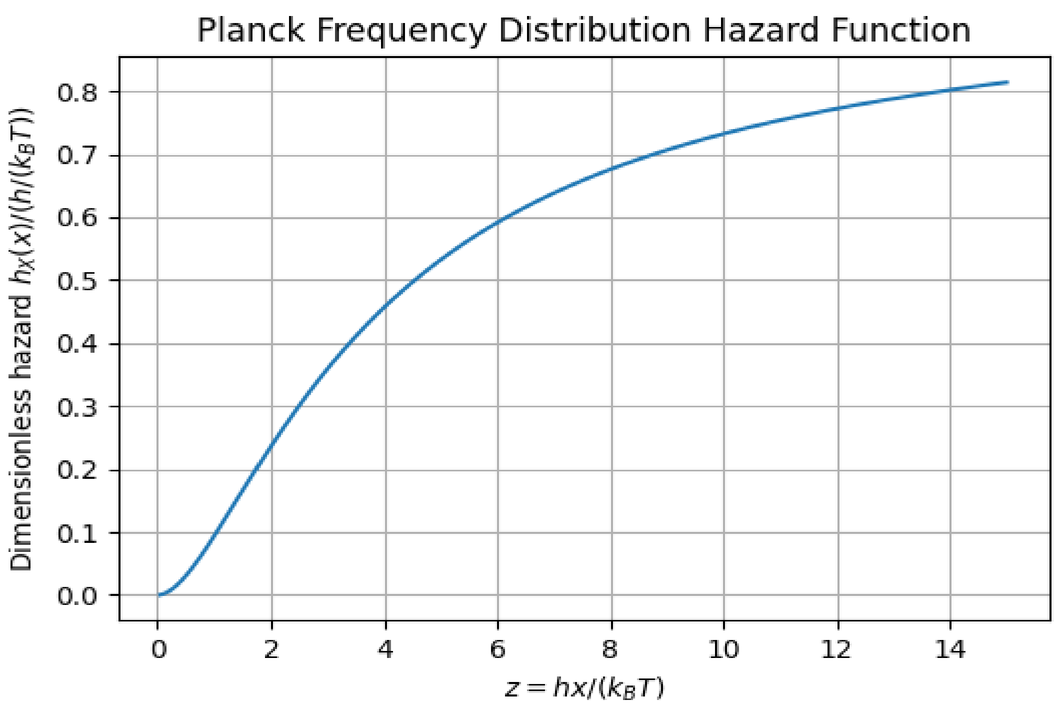

Figure 2: Planck's Hazard Function Plot

Shape Analysis of Planck's Hazard Function Plot: For small frequencies ($x \to 0$), $e^z - 1 \sim z$, and, therefore, we have

$$p(x) \sim \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^2.$$

Since $S(x) \to 1$, therefore we have

$$\lambda(x) \sim \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^3 x^2.$$

Thus, the hazard starts at zero and initially increases quadratically.

Further, for large frequencies ($x \to \infty$), since

$$p(x) \propto x^3 e^{-hx/(k_B T)}, \text{ and } S(x) \propto e^{-hx/(k_B T)},$$

it follows that

$$\lim_{x\to\infty} \lambda(x) = \frac{h}{k_B T}.$$

Therefore, from the above, we conclude the following:

(i) $\lambda(0) = 0$,

(ii) the hazard increases monotonically, and

(iii) it approaches the horizontal asymptote $h/(k_B T)$.

Furthermore, from Figure 2 of the Planck's Hazard Function Plot, we observe the following:

(a) it is increasing failure rate (IFR),

(b) a convex rise near the origin,

(c) a gradual flattening, and

(d) converges to the constant $h/(k_B T)$.

Thus, the Planck's probability density function produces a rising-to-plateau hazard curve, not a bathtub-shaped hazard function.

Effect of Temperature: From (5.17), since

$$\lambda(x) = \frac{h}{k_B T} M\left(\frac{hx}{k_B T}\right),$$

where $M(\cdot)$ is dimensionless, it follows that

I. increasing T shifts the hazard curve to the right,

II. increasing T lowers the asymptotic level $\frac{h}{k_B T}$,

III. decreasing T shifts the curve left and increases its limiting level.

Hence all temperature-dependent hazard curves are scaled versions of the same universal dimensionless shape.

Visual display of Planck's Mill's Ratio Plot and its Shape Analysis: Using some Python codes, we have plotted the Mill's ratio (5.19), in the following figure, Figure 3, for some selected special values of the frequency $\nu = x$, across some values of the temperature T, which provides a complete visual analysis and behavior of the physical Planck's Mill's ratio (5.19) curves.

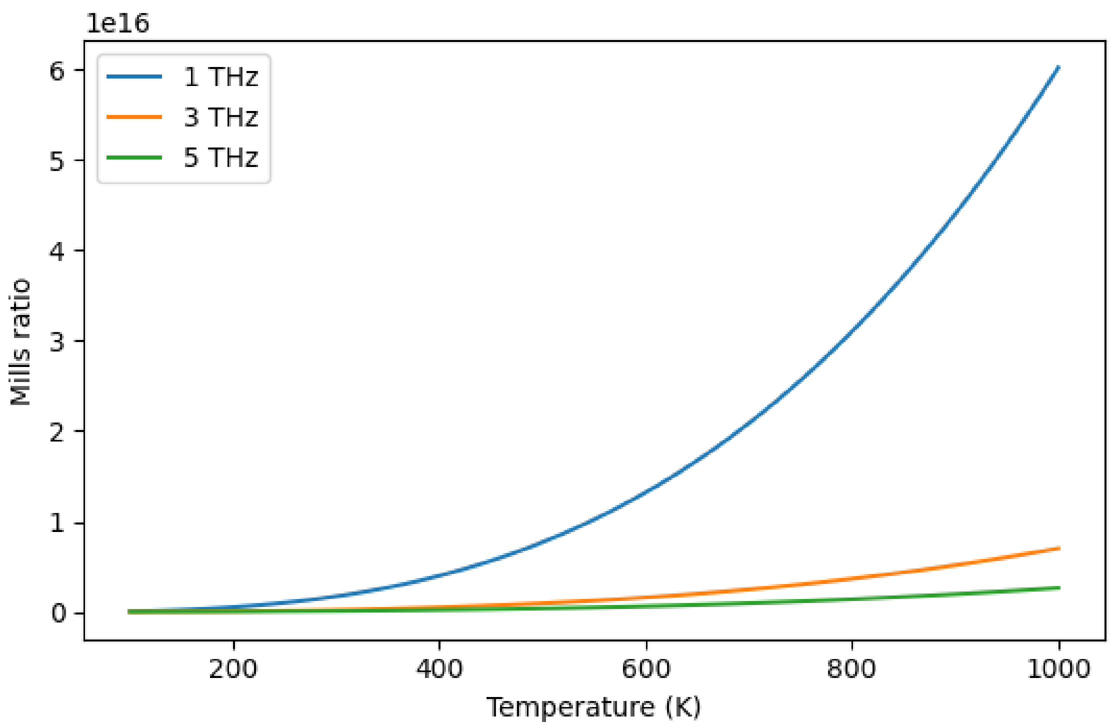

Figure 3: Mill's ratio versus temperature (100–1000 K) for selected frequencies ν = 1, 3 and 5 THz.

Obviously, as observed from Figure 3, for fixed frequency, the Mill's ratio generally decreases as temperature increases because the survival probability above the selected frequency declines relative to the local density. Lower frequencies produce larger Mill's ratios than higher frequencies over most of the temperature range, indicating a longer expected residual tail beyond lower frequencies. At higher frequencies the survival probability decays more rapidly, yielding smaller Mill's ratios. The curves become flatter at higher temperatures, reflecting the reduced sensitivity of the residual lifetime measure to further temperature increases.

6. BASIC STATISTICAL PROPERTIES

This section derives the expressions for the moments and the moment generating function, along with others, of the proposed Planck's probability distribution. All of these play a fundamental role in statistical theory and practical applications. The moments provide important insights into the distributional characteristics of the random variable, including its location, dispersion, asymmetry, and tail behavior, which are commonly described through the measures of central tendency, variability, skewness, and kurtosis.

A normalized Planck density defines a valid continuous probability distribution over the positive real line. Several important statistical properties of Planck's distribution may be derived analytically. In what follows, we derive these statistical properties of the exact physical Planck's continuous probability density function of frequency ν, as given by (5.13) above, where, without loss of generality, for the sake of convenience, replacing the frequency ν by x, let $p(x)$ denote a Planck's continuous probability density function of frequency x corresponding to a positive continuous random variable X. Then, we have

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}}-1}, \quad x > 0. \tag{6.1}$$

6.1 Moments and Moment Generating Function

6.1.1 rth Moment: The rth raw moment is defined as

$$E(X^r) = \int_0^\infty x^r p(x) dx$$

$$= \int_0^\infty x^r \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}}-1} dx. \tag{6.2}$$

Letting $u = \frac{hx}{k_B T} \Rightarrow x = \frac{k_B T}{h} u \Rightarrow dx = \frac{k_B T}{h} du$ in (6.2), and simplifying, we obtain

$$E(X^r) = \frac{15}{\pi^4} \left(\frac{k_B T}{h} \right)^r \int_0^\infty \frac{u^{r+3}}{e^u - 1} du. \tag{6.3}$$

For the integral on the right side of the equation (6.3), using the well-known the Bose-Einstein integral, which is defined in terms of the Gamma function (Γ) and the Riemann Zeta function (ζ) in (6.3), as follows (Gradshteyn and Ryzhik [46, 2007]):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(\mathrm{s})} \int_0^\infty \frac{z^{s-1}}{e^z - 1} dz,\ \mathcal{R}(s) > 1,$$

or,

$$\int_0^\infty \frac{z^{s-1}}{e^z - 1} dz = \Gamma(\mathrm{s})\, \zeta(s),\ \mathcal{R}(s) > 1, \tag{6.4}$$

where $\Gamma(\mathrm{s}) = \int_0^\infty z^{s-1} e^{-z} dz$, is the gamma function, the rth moment (6.3) can be evaluated explicitly given by

$$E(X^r) = \frac{15}{\pi^4} \left(\frac{k_B T}{h} \right)^r \Gamma(\mathrm{r}+4)\, \zeta(r+4). \tag{6.5}$$

6.2 Mean (First Moment)

For $r = 1$, the expression in (6.5) evaluates exactly to the following

$$E(X) = \frac{15}{\pi^4} \Gamma(5)\, \zeta(5) \frac{k_B T}{h}. \tag{6.6}$$

Since $\zeta(5) \approx 1.036927$ is a known value of the Riemann Zeta Function (Apostol [49, 1976, Chapter 12]; Titchmarsh [50, 1986, Chapter 1]), the evaluated value of gamma function $\Gamma(5) = 4! = 24$, and $\pi \approx 3.14159$, substituting these values in (6.6), it simplifies numerically to

$$E(X) \approx 3.8322 \frac{k_B T}{h}, \tag{6.7}$$

which is the required mean frequency (1st moment) of the Planck's continuous probability density function of frequency ν.

6.3 Second Moment

For $r = 2$, from (6.5) we obtain

$$E(X^2) = \frac{15}{\pi^4}\Gamma(6)\,\zeta(6)\left(\frac{k_B T}{h}\right)^2. \tag{6.8}$$

Since $\Gamma(6) = 5! = 120$ and $\zeta(6) = \frac{\pi^6}{945}$, (Apostol [49, 1976, Chapter 12]; Titchmarsh [50, 1986, Chapter 1]), using these values in (6.9), it simplifies to

$$\begin{aligned} E(X^2) &= \frac{40\,\pi^2}{21}\left(\frac{k_B T}{h}\right)^2 \\ &\approx 18.7992\left(\frac{k_B T}{h}\right)^2, \end{aligned} \tag{6.9}$$

which is the required 2nd moment of the Planck's continuous probability density function of frequency ν.

6.4 Variance

Variance measures the statistical spread of photon frequencies and is given by

$$Var(X) = \sigma^2 \; = \; E(X^2) - [E(X)]^2. \tag{6.10}$$

Using (6.7) and (6.9) in (6.10), we obtain

$$\begin{aligned} \sigma^2 &\approx \; 18.7992\left(\frac{k_B T}{h}\right)^2 - \left[3.8322\,\frac{k_B T}{h}\right]^2 \\ &\approx 4.1134\left(\frac{k_B T}{h}\right)^2, \end{aligned} \tag{6.11}$$

which is the required variance of the Planck's continuous probability density function of frequency ν.

6.5 Mode

In what follows, we derive the mode of the Planck's probability distribution with the pdf (6.1) given by

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{hx/(k_B T)} - 1}, x > 0,$$

where, without loss of generality, for the sake of convenience, the frequency ν has been replaced by x, from which on defining $a = \frac{h}{k_B T}$, we obtain

$$p(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax}-1}, x > 0.$$

Since mode is the value x_{mode} $(= x_m)$ that maximizes the density $p(x)$, and since the constant factor $\frac{15}{\pi^4} a^4$ does not affect the maximization, let us consider

$$f(x) = \frac{x^3}{e^{ax}-1} .$$

Differentiating both sides of the above equation with respect to x, we obtain

$$f'(x) = \frac{3x^2(e^{ax}-1) - ax^3 e^{ax}}{(e^{ax}-1)^2}.$$

Setting $f'(x) = 0$, since $(e^{ax} - 1)^2 > 0$, we obtain

$$3x^2(e^{ax} - 1) - ax^3 e^{ax} = 0.$$

Dividing both sides of the above equation by x^2, we obtain

$$3(e^{ax} - 1) - ax\, e^{ax} = 0.$$

Letting $y = ax$ in the above equation, we have

$$3(e^y - 1) - ye^y = 0,$$

or

$$(3 - y)e^y = 3,$$

which is a transcendental equation that determines the mode.

Now, letting $u = 3 - y$ in the above equation, we obtain

$$ue^{3-u} = 3,$$

or

$$ue^{-u} = 3e^{-3}.$$

Multiplying both sides of the above equation by -1, we obtain

$$(-u)e^{-u} = -3e^{-3}.$$

Now, in the above equation, we use the well-known Lambert W-function, denoted as $W(x)$, which is defined as the inverse of the function $f(W) = W \cdot e^W$, such that $W(x) \cdot e^{W(x)} = x$, or, equivalently, $W(x \cdot e^x) = x$, or $W^{-1}(x) = x \cdot e^x$, (see Corless *et al.* [51, 1996], Hassani [52, 2005], Brito *et al*. [53, 2008], Georg [54, 2011], and Dence [55, 2013], among others). It is worth mentioning here that the Lambert W function has found applications in many fields such as mathematics, physics, chemistry, biology, probability, and statistics. It also arises in asymptotic approximations, the analysis of algorithms, combinatorics, quantum physics, and risk theory, among others. Thus, we obtain the following Lambert W function representation of the above equation

$$-u = W(-3e^{-3}),$$

so that, substituting $u = 3 - y$, we obtain

$$y = 3 + W(-3e^{-3}).$$

Substituting $y = ax$, where $a = \frac{h}{k_B T}$, in the above equation, we obtain

$$x_{\text{mode}} = \frac{k_B T}{h}[3 + W(-3e^{-3})],$$

which is the required mode of Planck's probability density function in terms of Lambert W function. Moreover, after numerical calculations of the right-hand side of the above equation for the x_{mode}, involving Lambert W function, we obtain its principal real solution given by

$$3 + W(-3e^{-3}) \approx 2.8214,$$

which on substituting in the above equation for the x_{mode}, the numerical value of the x_{mode} ($= x_m$) is given by

$$x_m = x_{\text{mode}} \approx 2.8214\ \frac{k_B T}{h}.$$

6.6 Median

Here we shall we derive the median of the Planck's probability distribution by considering its pdf (6.1) in the following form

$$p(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax}-1}, x > 0 ,$$

where $a = \frac{h}{k_B T}$.

Now, recalling that the median, say, m, of a probability distribution of a positive continuous random variable X satisfies the following inequality

$$P(X \leq m) = \frac{1}{2} .$$

Thus, for the Planck's probability distribution with the above pdf, we have

$$\frac{15}{\pi^4} a^4 \int_0^m \frac{x^3}{e^{ax}-1}\, dx = \frac{1}{2}.$$

Then, letting $y = ax,\ M = am$, in the integral of the above median equation, we obtain

$$\int_0^m \frac{x^3}{e^{ax}-1}\, dx = \frac{1}{a^4} \int_0^M \frac{y^3}{e^{y}-1}\, dy,$$

which, on substituting back into the above median equation, after simplification, we obtain

$$\int_0^M \frac{y^3}{e^{y}-1}\, dy = \frac{\pi^4}{30}.$$

from which, on using the definition of the nth order Debye function given by

$D_n(z) = \frac{n}{z^n} \int_0^z \frac{t^n}{e^t-1} dt$, (Gradshteyn and Ryzhik [47, 2007]; Abramowitz and Stegun [48, 1972]), when $n = 3$, after simplification, we obtain the following equation in terms of the third-order Debye function $D_3(M)$:

$$D_3(M)\, M^3 = \frac{\pi^4}{10} ,$$

where $M = am$, and $a = \frac{h}{k_B T}$, and therefore, the median m is the unique positive solution of the following equation

$$m^3 D_3\left(\frac{h}{k_B T} m\right) = \frac{\pi^4}{10}\left(\frac{k_B T}{h}\right)^3.$$

Now the equation $D_3(M)\, M^3 = \frac{\pi^4}{10}$ can be solved numerically for M using the software Python (via the SciPy library) or MATLAB or Wolfram Mathematica / WolframAlpha. Thus, solving it numerically using the Python, we obtain the following numerical value of M:

$$M \approx 3.50395.$$

Since $M = am$, we have

$$m = \frac{M}{a} = \frac{k_B T}{h} M.$$

Therefore, for $M \approx 3.50395$, we obtain

$$m = x_{\text{median}} \approx 3.50395\ \frac{k_B T}{h}.$$

6.7 Moment Generating Function (MGF)

For a positive continuous random variable X, from (6.1), replacing the frequency ν by x, we have the Planck's probability density function of frequency x given by

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{hx/(k_B T)} - 1}, x > 0,$$

$$where\ h = \ Planck\ constant, k_B = \ Boltzmann\ constant, and\ T > 0 = \ absolute\ temperature.$$

Let

$$a = \frac{h}{k_B T}.$$

Then

$$p(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax} - 1}, x > 0.$$

For any random variable X, with the nth moment $E(X^n)$, the moment generating function is given by

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n), \tag{6.12}$$

provided the series converges.

For the Planck's frequency distribution, from (6.5), we have

$$E(X^n) = \frac{15}{\pi^4}\Gamma(n+4)\,\zeta(n+4)\left(\frac{k_B T}{h}\right)^n,$$

where $\zeta(n+4)$ denotes the Riemann zeta function.

Substituting for $E(X^n)$ into the MGF series (6.12), we obtain

$$M_X(t) = \frac{15}{\pi^4}\sum_{n=0}^{\infty}\frac{\Gamma(n+4)\zeta(n+4)}{n!}\left(\frac{k_B T}{h}t\right)^n.$$

from which, on using $\Gamma(n+4) = (n+3)!$, we obtain

$$M_X(t) = \frac{15}{\pi^4}\sum_{n=0}^{\infty}\frac{(n+3)!}{n!}\zeta(n+4)\left(\frac{k_B T}{h}t\right)^n. \tag{6.13}$$

Now, using the definition of the Riemann zeta function, we have

$$\zeta(n+4) = \sum_{m=1}^{\infty}\frac{1}{m^{n+4}}\,.$$

Therefore, substituting it in (6.13), we obtain

$$M_X(t) = \frac{15}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^4}\sum_{n=0}^{\infty}\frac{(n+3)!}{n!}\left(\frac{k_B T}{hm}t\right)^n. \tag{6.14}$$

Thus, for the inner series in (6.14), using the following well-known standard generating-function identity

$$\sum_{n=0}^{\infty}\frac{(n+3)!}{n!}z^n = \frac{6}{(1-z)^4},\ \mid z \mid < 1,$$

we obtain

$$M_X(t) = \frac{90}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{\left(m - \frac{k_B T}{h}t\right)^4}. \tag{6.15}$$

Hence, using the Hurwitz zeta function (see, for example, Gradshteyn and Ryzhik [47, 2007]; Magnus *et al*. [56, 1966]; Prudnikov *et al*. [57, 1990]; and Olver *et al*. [58, 2010]) in (6.15), which is defined as follows

$$\zeta(s,q) = \sum_{m=0}^{\infty} \frac{1}{(m+q)^s},$$

we obtain

$$M_X(t) = \frac{90}{\pi^4}\zeta\left(4,1-\frac{k_BT}{h}t\right), t < \frac{h}{k_BT}, \tag{6.16}$$

which is the required moment generating function (MGF) of Planck's probability density function.

Remark 6.7.1:

From (6.13), we have

$$M_X(t) = \frac{15}{\pi^4}\sum_{n=0}^{\infty} \frac{(n+3)!}{n!}\zeta(n+4)\left(\frac{k_BT}{h}t\right)^n$$

$$= \frac{15}{\pi^4}\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)\zeta(n+4)\left(\frac{k_BT}{h}t\right)^n,$$

from which, after numerical calculations, the first few terms are easily obtained as follows:

$$M_X(t) = 1 + \frac{360\zeta(5)}{\pi^4}\frac{k_BT}{h}t + \frac{1800\zeta(6)}{\pi^4}\left(\frac{k_BT}{h}\right)^2 t^2 + \cdots$$

or

$$M_X(t) = 1 + E(X)t + \frac{E(X^2)}{2!}t^2 + \cdots .$$

It is easily verified from above that

$$M_X(0) = \frac{90}{\pi^4}\zeta(4) = \frac{90}{\pi^4}\cdot\frac{\pi^4}{90} = 1.$$

Moreover, we have

$$M_X'(0) = E(X) = \frac{360\zeta(5)}{\pi^4}\frac{k_BT}{h}.$$

Thus the MGF (6.16) is fully consistent with the first moment.

Remark 6.7.2: Raw Moments of the Planck Frequency Distribution

Recall that

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_BT}\right)^4 \frac{x^3}{e^{\frac{hx}{k_BT}}-1}, \quad x > 0,$$

and

$$\mu_r' = E(X^r) = \frac{15}{\pi^4}\left(\frac{k_B T}{h}\right)^r \Gamma(r+4)\zeta(r+4).$$

Now, we define the scale parameter as

$$\theta = \frac{k_B T}{h}.$$

Then, we have

$$\mu_r' = \frac{15}{\pi^4}\Gamma(r+4)\zeta(r+4)\theta^r. \tag{6.17}$$

Hence, from (6.17), the first four raw moments, in terms of the Riemann Zeta function, are obtained as follows:

$$\mu_1' = \frac{360\,\zeta(5)}{\pi^4}\theta,$$

$$\mu_2' = \frac{1800\,\zeta(6)}{\pi^4}\theta^2,$$

$$\mu_3' = \frac{10800\,\zeta(7)}{\pi^4}\theta^3,$$

and

$$\mu_4' = \frac{75600\,\zeta(8)}{\pi^4}\theta^4.$$

Furthermore, it is well known that the Riemann Zeta function has exact, closed-form numerical values in terms of π for all positive even integers. These formulas were discovered by Leonhard Euler and rely on the Bernoulli numbers. For odd integers, no known closed-form formula involving π exists, and these values must be approximated numerically. For any positive even integer $2n$, the values of the Riemann Zeta function in terms of π are given by the following general formula:

$$\zeta(2n) = (-1)^{n+1}\,\frac{(2\pi)^{2n}\,B_{2n}}{2\,(2n)!}, \tag{6.18}$$

where B_{2n} represents the Bernoulli numbers ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, etc.). Thus, from (6.18), we have $\zeta(6) = \frac{\pi^6}{945}$, and $\zeta(8) = \frac{\pi^8}{9450}$. For details, see, for example, Apostol [49, 1976, Chapter 12], and Titchmarsh [50, 1986, Chapter 1], among others. Hence, using these values, we obtain

$$\mu_2' = \frac{40\pi^2}{21}\theta^2,$$

and

$$\mu_4' = 8\pi^4\theta^4.$$

6.8 Characteristic Function (CF)

The characteristic function (CF) of Planck's probability density function is given by

$$\phi_X(\omega) = E(e^{i\omega X}) = M_X(i\omega). \tag{6.19}$$

Therefore, using the expression (6.16) of the MGF in (6.19, we obtain

$$\phi_X(\omega) = \frac{90}{\pi^4}\zeta\left(4,1 - i\frac{k_B T}{h}\omega\right), \tag{6.20}$$

which is the required characteristic function (CF) of Planck's probability density function, in terms of the Hurwitz zeta function $\zeta(s,q) = \sum_{m=0}^{\infty} \frac{1}{(m+q)^s}$.

6.9 Skewness

To find the skewness of Planck's probability density function, we proceed as follows: The third central moment is given by

$$\mu_3 = \mu_3' - 3\mu\mu_2' + 2\mu^3. \tag{6.21}$$

Substituting in (6.21) the raw moments as obtained in Remark 6.7.2 above, we obtain

$$\mu_3 = \theta^3\left[\frac{10800\zeta(7)}{\pi^4} - \frac{43200}{21}\zeta(5) + 2\left(\frac{360\zeta(5)}{\pi^4}\right)^3\right]. \tag{6.22}$$

From (6.22), after numerical calculations, we obtain

$$\mu_3 \approx 13.8463\,\theta^3. \tag{6.23}$$

Now, the coefficient of skewness is given by

$$\gamma_1 = \frac{\mu_3}{\sigma^3}, \tag{6.24}$$

where, from (6.11), we have $\sigma \approx 2.02820\,\theta$, where $\theta = \frac{k_B T}{h}$.

Hence, substituting for μ_3 and σ in (6.24), after numerical calculations, we obtain

$$\gamma_1 \approx 1.659. \quad (6.25)$$

It is obvious from (6.25) that

(i) $\gamma_1 > 0$;

(ii) Planck's probability distribution has a strong right tail;

(iii) Planck's probability distribution is consistent with the exponential decay of Planck's radiation law.

Thus, the Planck's probability distribution is positively skewed.

6.10 Kurtosis

To find the kurtosis, we recall that the fourth central moment is given by

$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4,$$

from which, on substituting the raw moments as obtained in the Remark 6.5.2 above, and, after numerical calculations, we obtain

$$\mu_4 \approx 95.53\,\theta^4.$$

Now, the kurtosis coefficient is given by

$$\beta_2 = \frac{\mu_4}{\sigma^4}. \quad (6.26)$$

Hence, substituting for μ_4 and σ in (6.26), after numerical calculations, we obtain

$$\beta_2 \approx 5.65. \quad (6.27)$$

Hence, from (6.27), the excess kurtosis is obtained as follows:

$$\gamma_2 = \beta_2 - 3 \approx 2.65. \quad (6.28)$$

It is obvious from (6.27) that, since $\beta_2 > 3$, the Planck's probability distribution is leptokurtic. Also, it is observed that it has

(i) heavier tails than a normal distribution;

(ii) a sharper peak;

(iii) greater probability of extreme frequencies.

6.11 Shannon Entropy

The Maximum Entropy Principle (MEP) provides a systematic framework for constructing probability distributions when only partial information about a random variable is available. According to this principle, the preferred distribution is the one that maximizes entropy while satisfying the prescribed constraints. As a result, the selected distribution remains fully consistent with the known information without incorporating any unsupported assumptions or subjective bias. Owing to this property, since its introduction by Jaynes in 1957, the MEP has become an important tool in information theory, statistical inference, and statistical mechanics (Jaynes [59, 1957]; Kapur [60, 1989]).

The notion of entropy for an absolutely continuous random variable (X) was first introduced by Shannon in 1948 as a quantitative measure of uncertainty associated with a probability distribution (Shannon [61, 1948]). Entropy reflects the extent of randomness or unpredictability in the possible outcomes of a random variable and measures the information gained from an observation about its underlying distribution. In general, distributions with higher entropy correspond to greater uncertainty, whereas lower entropy indicates that the outcomes are more predictable.

In what follows, we derive the Shannon differential entropy of Planck's continuous probability distribution with the pdf (6.1) given by

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{hx/(k_B T)}-1}, x > 0,$$

where, without loss of generality, for the sake of convenience, the frequency ν has been replaced by x, from which on defining $a = \frac{h}{k_B T}$, we obtain

$$p(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax}-1}, x > 0.$$

Then, letting $C = \frac{15}{\pi^4} a^4$, we have

$$p(x) = C\,\frac{x^3}{e^{ax}-1}\,.$$

Now, for a positive continuous random variable X, the Shannon entropy is given by

$$H(X) = -\int_0^\infty p(x) \ln p(x)\, dx. \tag{6.29}$$

Substituting for $p(x)$ in (6.29), we obtain

$$H(X) = -\int_0^\infty p(x)[\ln C + 3\ln x - \ln(e^{ax} - 1)]dx. \tag{6.30}$$

Since $\int_0^\infty p(x)\, dx = 1$, from (6.30), we obtain

$$H(X) = -\ln C - 3E(\ln X) + E[\ln(e^{aX} - 1)], \tag{6.31}$$

where $E(\ln X)$ and $E[\ln(e^{aX} - 1)$ are evaluated as follows:

<u>Evaluation of $E(\ln X)$</u>

Let $Y = aX$, where the random variable X has the Planck's probability density function as above given by $p(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax}-1}, x > 0.$ Applying the Jacobian method $f_Y(y) = f_Y(y)\left|\frac{dx}{dy}\right|$ to $Y = aX$, it is easily seen that the scale parameter a cancels out. Thus, the resulting probability density function of Y becomes

$$g(y) = \frac{15}{\pi^4} \frac{y^3}{e^y - 1}, y > 0.$$

Hence

$$E(\ln X) = -\ln a + E(\ln Y).$$

Now

$$E(\ln Y) = \frac{15}{\pi^4} \int_0^\infty \frac{y^3 \ln y}{e^y - 1}\, dy.$$

Now, recalling $\displaystyle\int_0^\infty \frac{y^{s-1}}{e^y - 1}\, dy = \Gamma(s)\zeta(s)$, (Gradshteyn and Ryzhik [47, 2007, 3.411.1, page 325]), defined in terms of the Gamma function (Γ) and the Riemann Zeta function (ζ), from which on differentiating both sides with respect to s under the integral sign, we obtain

$$\int_0^\infty \frac{\partial}{\partial s}\left(\frac{y^{s-1}}{e^y - 1}\right) dy = \frac{d}{ds}(\Gamma(s)\zeta(s)),$$

or

$$\int_0^\infty \frac{y^{s-1}\ln y}{e^y-1}\,dy = \Gamma(s)\zeta(s)\left[\psi(s)+\frac{\zeta'(s)}{\zeta(s)}\right].$$

Substituting $s = 4$ in the above equation, we obtain

$$E(\ln Y) = \psi(4) + \frac{\zeta'(4)}{\zeta(4)},$$

which on using in the above-mentioned expression for $E(\ln X)$, we obtain

$$E(\ln X) = -\ln a + \psi(4) + \frac{\zeta'(4)}{\zeta(4)}. \tag{6.32}$$

Evaluation of $E[\ln(e^{aX}-1)]$

Again using $Y = aX$, we obtain

$$E[\ln(e^{aX}-1)] = \frac{15}{\pi^4}\int_0^\infty \frac{y^3\ln(e^y-1)}{e^y-1}\,dy.$$

Since $\ln(e^y-1) = y + \ln(1-e^{-y})$, we obtain

$$E[\ln(e^{aX}-1)] = \frac{15}{\pi^4}\left(\int_0^\infty \frac{y^4}{e^y-1}\,dy + \int_0^\infty \frac{y^3\ln(1-e^{-y})}{e^y-1}\,dy\right).$$

Now, using $\int_0^\infty \frac{y^{s-1}}{e^y-1}\,dy = \Gamma(s)\zeta(s)$ in the above equation, the first integral is given by

$$\int_0^\infty \frac{y^4}{e^y-1}\,dy = 24\zeta(5).$$

For the second integral $\int_0^\infty \frac{y^3\ln(1-e^{-y})}{e^y-1}\,dy$, we note that

$$\ln(1-e^{-y}) = -\sum_{m=1}^\infty \frac{e^{-my}}{m}.$$

Therefore, using formula 3.411.6, page 325, of Gradshteyn and Ryzhik [47, 2007], we obtain

$$\int_0^\infty \frac{y^3\ln(1-e^{-y})}{e^y-1}\,dy = -6\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{1}{m(n+m)^4}.$$

Now, recalling the well-known Euler sum identities or harmonic double series, (Olver *et al*. [58, 2020]; Tornheim [62, 1950]; Hoffman [63, 1992]), defined as

$$\sum_{n,m\geq 1} \frac{1}{m(n+m)^p} = \zeta(p),$$

where ζ denotes the Riemann zeta function, we obtain

$$\sum_{n,m\geq 1} \frac{1}{m(n+m)^4} = \zeta(5).$$

Therefore, in view of the above, the second integral reduces to

$$\int_0^\infty \frac{y^3 \ln(1-e^{-y})}{e^y - 1} dy = -6\zeta(5).$$

Consequently, we have

$$E[\ln(e^{aX} - 1)] = \frac{15}{\pi^4}(24\zeta(5) - 6\zeta(5)) = \frac{270}{\pi^4}\zeta(5). \tag{6.33}$$

Thus, combining the above expressions (6.32) and (6.33) for the $E(\ln X)$ and $E[\ln(e^{aX} - 1)$, respectively, in (6.31), we obtain

$$H(X) = -\ln\left(\frac{15}{\pi^4}a^4\right) - 3\left[-\ln a + \psi(4) + \frac{\zeta'(4)}{\zeta(4)}\right] + \frac{270}{\pi^4}\zeta(5),$$

from which, after simplification, we obtain

$$H(X) = \ln\left(\frac{\pi^4}{15}\right) - \ln a - 3\psi(4) - 3\frac{\zeta'(4)}{\zeta(4)} + \frac{270}{\pi^4}\zeta(5),$$

and, since $a = h/(k_B T)$, we have

$$H(X) = \ln\left(\frac{\pi^4}{15}\frac{k_B T}{h}\right) - 3\psi(4) - 3\frac{\zeta'(4)}{\zeta(4)} + \frac{270}{\pi^4}\zeta(5),$$

which is the required Shannon entropy of Planck's continuous probability distribution with the pdf (6.1). Here $k_B$is the Boltzmann constant, h is the Planck constant, $\psi(\cdot)$ is the digamma function, $\zeta(\cdot)$ is the Riemann zeta function, and T is measured in kelvin. In the above expression for the Shannon entropy, we also observed that temperature T is the only variable in this expression; all other terms are constant. Hence, the above Shannon entropy is a function of T only, which can be expressed as

$$H(T) = C + \ln T,$$

where

$$C = \ln\left(\frac{\pi^4 k_B}{15h}\right) - 3\psi(4) - 3\frac{\zeta'(4)}{\zeta(4)} + \frac{270}{\pi^4}\zeta(5).,$$

Furthermore, using the following computed numerical values of the Gamma function (Γ) and the Riemann Zeta function (ζ), (Gradshteyn and Ryzhik [47, 2007]; Olver *et al*. [58, 2020]),

$\psi(4) \approx 1.256117668,$

$\zeta(5) \approx 1.036927755,$ and

$\frac{\zeta'(4)}{\zeta(4)} \approx -0.063669,$

we obtain the following numerical form of the Shannon entropy of Planck's continuous probability distribution

$$H(X) \approx 1.25569 + \ln\left(\frac{k_B T}{h}\right),$$

that is,

$$H(T) \approx 12.7926 + \ln T,$$

where $T$is measured in kelvin.

Visual display of Planck's Shannon Entropy Plot and its Shape Analysis: For some selected special values of the representative temperatures (100–1000 K), using some Python codes to generate precise numerical values and visual graph of the above Shannon entropy of Planck's continuous probability distribution, we have provided in the following Table 6.1, and sketched the corresponding figure, Figure 6.1, which provide a complete visual analysis and behavior of the Shannon entropy of Planck's continuous probability distribution.

Table 6.1: Numerical Values of Shannon entropy $H(T)$

T(K)	Shannon entropy $H(T)$
100	17.40
200	18.09
300	18.50
400	18.78
500	19.01
600	19.19
800	19.48
1000	19.70

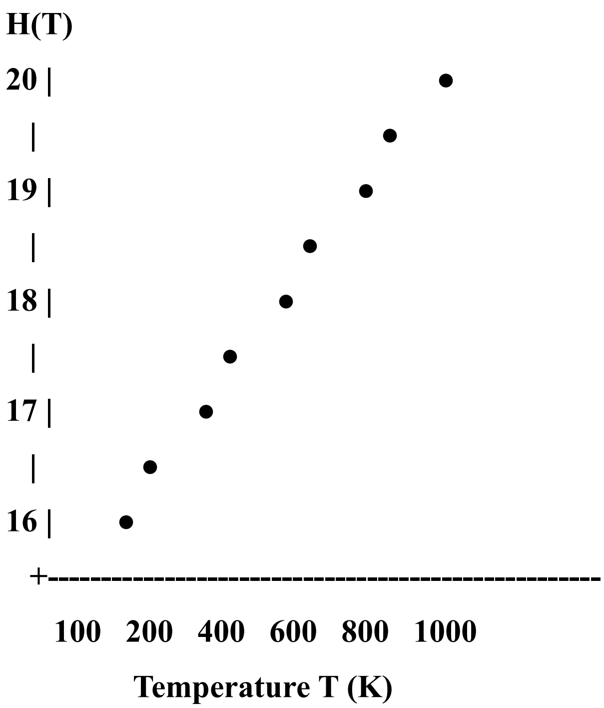

Figure 6.1: Graph Shannon entropy $H(T)$ for representative temperatures (100–1000 K)

It is observed from Figure 6.1 and Table 6.1 that the Shannon entropy increases logarithmically with temperature, and also it is a monotonically increasing logarithmic function of the absolute temperature. At low temperatures, the increase in entropy is relatively rapid, whereas at higher temperatures the rate of increase gradually diminishes because of the logarithmic dependence. Physically, this indicates that as the temperature rises, the Planck distribution becomes more dispersed, leading to greater uncertainty or randomness in the energy of a randomly selected photon. Consequently, the information content (or uncertainty) associated with the distribution increases with temperature, although the increase occurs at a progressively slower rate. Moreover, this logarithmic behavior is consistent with the thermodynamic interpretation of entropy and demonstrates that the Shannon entropy of Planck's distribution grows steadily with increasing thermal energy.

7. Characterization

Characterization of probability distributions aims to establish unique properties or conditions that distinguish a given distribution from all others. As emphasized by Nagaraja [64, 2006] and Koudou and Ley [65, 2014], characterization plays a fundamental role in understanding stochastic models and fostering the development of new mathematical theories. Significant contributions to the characterization of probability distributions have been made by Galambos and Kotz [66, 1978], Glänzel *et al*. [67, 1984], Ahsanullah [68, 2017], and Shakil *et al*. ([46, 2025; 69, 2018; 70, 2021]), among many others. In this section, we focus on the characterization of Planck's continuous probability distribution with the pdf given in (6.1) using truncated moments.

7.1. Characterization by Truncated Moments

In this subsection, we present two new characterization results for Planck's continuous probability distribution with the pdf (6.1) based on truncated moments. The first characterization result (Theorem 7.1) is established through a relationship between the left-truncated moment and the failure rate function. The second characterization result (Theorem 7.2) is derived from a relationship between the right-truncated moment and the reversed failure rate function. To develop these results, we first require the following proposition, assumption and lemmas, which can be found, for example, in Shakil *et al*. ([46, 2025; 69, 2018; 70, 2021]).

Proposition 7.1. *If the random variable X has Planck's continuous probability distribution with the pdf (6.1) given by*

$$p(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{hx/(k_B T)} - 1}, x > 0,$$

where, without loss of generality, for the sake of convenience, the frequency ν *has been replaced by* x, *from which, on defining* $a = \frac{h}{k_B T}$ *and letting* $C = \frac{15}{\pi^4} a^4$, *we obtain*

$$p(x) = C \, \frac{x^3}{e^{ax-1}}, x > 0.$$

Then its jth *incomplete moment,* $I_j(x)$, *is given by*

$$I_j(x) = \frac{1}{j+3} \frac{15}{\pi^4} \left(\frac{h}{k_B T}\right)^3 (x)^{j+3} D_{j+3}\left(\frac{h}{k_B T} x\right), \quad \mathrm{x} > 0. \tag{25}$$

Proof. The jth incomplete moment's definition implies

$$I_j(x) = \int_0^x t^j \left[C \, \frac{t^3}{e^{at-1}}\right] dt. \tag{26}$$

Substituting $at = \mathrm{u}$ in (26), we obtain

$$I_j(x) = \int_0^{ax} \left(\frac{u}{a}\right)^j \left[C \, \frac{\left(\frac{u}{a}\right)^3}{e^{u-1}}\right] \frac{du}{a} = C \left(\frac{1}{a}\right)^{j+4} \int_0^{ax} \frac{(u)^{j+3}}{e^{u-1}} du,$$

and using the definition of the nth order Debye function given by

$D_n(z) = \frac{n}{z^n} \int_0^z \frac{t^n}{e^{t-1}} dt$, (Gradshteyn and Ryzhik [47, 2007]; Abramowitz and Stegun [48, 1972]), when $n = j + 3$**,** after simplification**,** we obtain the jth incomplete moment of X ~Planck's distribution as follows

$$I_j(x) = \frac{1}{j+3} \frac{C}{a} (x)^{j+3} D_{j+3}(ax).$$

Consequently, since $a = \frac{h}{k_B T}$ and $C = \frac{15}{\pi^4} a^4$, we obtain

$$I_j(x) = \frac{1}{j+3} \frac{15}{\pi^4} \left(\frac{h}{k_B T}\right)^3 (x)^{j+3} D_{j+3}\left(\frac{h}{k_B T} x\right), \quad \mathrm{x} > 0, \tag{27}$$

which is the required jth incomplete moment of X ~Planck's distribution.

Assumption 7.1. *Assume that the random variable* X *is absolutely continuous with CDF* $F(x)$, *and PDF* $f(x)$. *Let* $\omega = inf\{x | F(x) > 0\}$, *and* $\delta = sup\{x \mid F(x) < 1\}$, *represent the infimum and supremum of the support, respectively. Additionally,* $f(x)$ *is differentiable for all* x *and the expected value* $E(X)$ *is finite.*

Lemma 7.1. *If the random variable X satisfies the Assumption 7.1 with* $\omega = 0$ *and* $\delta = \infty$, *and if* $E(X|X \leq x) = g(x)\tau(x)$, *where* $\tau(x) = \frac{f(x)}{F(x)}$ *and* $g(x)$ *is a continuous differentiable function of* x *with the condition that* $\int_0^x \frac{u - g^{/}(u)}{g(u)} du$ *is finite, then* $f(x) = ce^{\int_0^x \frac{u - g^{/}(u)}{g(u)} du}$, *where* c *is a constant determined by the condition* $\int_0^\infty f(x)\, dx = 1$.

Proof. See Shakil *et al*. [69, 2018].

Lemma 7.2: *If the random variable* X *satisfies the Assumption 7.1 with* $\omega = 0$ *and* $\delta = \infty$, *and if* $E(X|X \geq x) = \tilde{g}(x)r(x)$, *where* $r(x) = \frac{f(x)}{1 - F(x)}$ *and* $\tilde{g}(x)$ *is a continuous differentiable function of* x *with the condition that* $\int_x^\infty \frac{u + [\tilde{g}(u)]^{/}}{\tilde{g}(u)} du$ *is finite, then* $f(x) = ce^{-\int_x^\infty \frac{u + [\tilde{g}(u)]^{/}}{\tilde{g}(u)} du}$, *where* c *is a constant determined by the condition* $\int_0^\infty f(x)\, dx = 1$.

Proof. See Shakil *et al*. [69, 2018].

Theorem 7.1: *If the random variable X satisfies Assumption 7.1 with* $\omega = 0$ *and* $\delta = \infty$, *then* $E(X|X \leq x) = g(x)\, \frac{f(x)}{F(x)}$, *where*

$$g(x) = \frac{1}{j+3} \frac{k_B T}{h} (x)^j \left(e^{hx/(k_B T)} - 1\right) D_{j+3}\left(\frac{h}{k_B T} x\right), x > 0, \tag{28}$$

if and only if X has the pdf $p(x)$, as given above in Proposition 7.1, which, for convenience, we replace by $p(x) = f(x)$, so that

$$f(x) = \frac{15}{\pi^4} \left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}} - 1}, x > 0,$$

or

$$f(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax} - 1}, x > 0, \tag{29}$$

where $a = \frac{h}{k_B T}$.

Proof. Suppose that $E(X|X \leq x) = g(x)\frac{f(x)}{F(x)}$. Then, since $E(X|X \leq x) = \frac{\int_0^x u\, f(u)\, du}{F(x)}$, we have $g(x) = \frac{\int_0^x u\, f(u)\, du}{f(x)}$. Now, if the random variable X satisfies Assumption 7.1 and has the distribution with the pdf (29), then, using (27), after simplification, we obtain

$$g(x) = \frac{\int_0^x u\, f(u)\, du}{f(x)} = \frac{1}{j+3}\,\frac{k_B T}{h}\,(x)^j\left(e^{hx/(k_B T)} - 1\right) D_{j+3}\left(\frac{h}{k_B T}x\right),\ \mathrm{x} > 0.$$

Consequently, the proof of "if" part of the Theorem 7.1 follows from Lemma 7.1.

Conversely, suppose that

$$g(x) = \frac{1}{j+3}\,\frac{k_B T}{h}\,(x)^j\left(e^{hx/(k_B T)} - 1\right) D_{j+3}\left(\frac{h}{k_B T}x\right),\mathrm{x} > 0. \tag{30}$$

Now, from Lemma 7.1, we obtain

$$g(x) = \frac{\int_0^x u\, f(u)\, du}{f(x)},$$

or

$$\int_0^x u\, f(u)\, du = f(x)\, g(x). \tag{31}$$

Differentiating the equation (31) with respect to x, and using the Fundamental Theorem of Calculus, we obtain

$$xf(x) = f^{/}(x)g(x) + f(x)g^{/}(x),$$

or

$$\frac{x - g^{/}(x)}{g(x)} = \frac{f^{/}(x)}{f(x)}\,. \tag{32}$$

Now, the derivative of the pdf (29) with respect to x is given by

$$f'(x) = \frac{15}{\pi^4}a^4\,\frac{3x^2(e^{ax}-1) - ax^3 e^{ax}}{(e^{ax}-1)^2}\,,$$

from which, using it along with the pdf (29), after simplification, we obtain

$$\frac{f^{/}(x)}{f(x)} = \frac{\frac{15}{\pi^4}a^4\frac{3x^2(e^{ax}-1) - ax^3 e^{ax}}{(e^{ax}-1)^2}}{\frac{15}{\pi^4}a^4\frac{x^3}{e^{ax}-1}} = \frac{3}{x} - \frac{a\,e^{ax}}{e^{ax}-1}\,. \tag{33}$$

Now, since, by Lemma 7.1, we have

$$\frac{x - g^{/}(x)}{g(x)} = \frac{f^{/}(x)}{f(x)} \text{ (Shakil \textit{et al}. [69, 2018]).} \tag{34}$$

Therefore, from (33) and (34), we obtain

$$\frac{f^{/}(x)}{f(x)} = \frac{3}{x} - \frac{a\,e^{ax}}{e^{ax}-1},$$

or,

$$\frac{d\,ln\big(f(x)\big)}{dx} = \frac{3}{x} - \frac{a\,e^{ax}}{e^{ax}-1}. \tag{35}$$

Integrating (35) with respect to x and simplifying, we obtain

$$ln\big(f(x)\big) = ln\left(\mathrm{c}\frac{x^3}{e^{a\,x}-1}\right),$$

or

$$f(x) = \mathrm{c}\frac{x^3}{e^{a\,x}-1}, \tag{36}$$

where c denotes the normalizing constant.

Integrating (36) with respect to x from $x = 0$ to $x = \infty$, and using the condition $\int_0^{\infty} f(x)\,dx = 1$, we obtain

$$\frac{1}{c} = \int_0^{\infty} \frac{x^3}{e^{a\,x}-1}\,dx. \tag{37}$$

For the integral on the right side of the equation (37), substituting $u = ax$ and using the well-known the Bose-Einstein integral, which is defined in terms of the Gamma function (Γ) and the Riemann Zeta function (ζ), (Gradshteyn and Ryzhik [47, 2007, Equation 3.411.1]), as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(\mathrm{s})} \int_0^{\infty} \frac{z^{s-1}}{e^z-1} dz,\ \mathcal{R}(s) > 1,$$

or

$$\int_0^{\infty} \frac{z^{s-1}}{e^z-1} dz = \Gamma(\mathrm{s})\,\zeta(s),\ \mathcal{R}(s) > 1,$$

where $\Gamma(\mathrm{s}) = \int_0^{\infty} z^{s-1} e^{-z} dz$, is the gamma function,), we obtain

$$\frac{1}{c} = \frac{1}{a^4}\Gamma(4)\,\zeta(4).$$

Since $\Gamma(4) = 3! = 6$ and $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, (Apostol [49, 1976, Chapter 12]; Titchmarsh [50, 1986, Chapter 1]), using these values in (6.9), the above expression for the normalizing constant simplifies to

$$c = \frac{15}{\pi^4} a^4,$$

from which, using it in (36), we obtain

$$f(x) \;=\; \frac{15}{\pi^4} a^4 \frac{x^3}{e^{a\,x} - 1},\; x > 0,$$

where $a = \frac{h}{k_B T}$. Thus,

$$f(x) = \frac{15}{\pi^4} \left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}} - 1},\; x > 0,$$

which is the required PDF of X ~Planck's distribution. This completes the proof of Theorem 7.1.

Theorem 7.2: *If the random variable X satisfies Assumption 7.1 with* $\omega = 0$ *and* $\delta = \infty$*, then* $E(X|X \ge x) = \tilde{g}(x) \frac{f(x)}{1 - F(x)}$*, where* $\tilde{g}(x) = \frac{(E(X) - g(x) f(x))}{f(x)}$*, where* $g(x)$ *is given by (28) and* $E(X)$ *is given by (6.6), if and only if X has the pdf*

$$f(x) = \frac{15}{\pi^4} \left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}} - 1}, x > 0,$$

or

$$f(x) = \frac{15}{\pi^4} a^4 \frac{x^3}{e^{ax} - 1}, x > 0, \tag{38}$$

where $a = \frac{h}{k_B T}$.

Proof. Suppose that $E(X|X \ge x) = \tilde{g}(x) \frac{f(x)}{1 - F(x)}$. Then, since $E(X|X \ge x) = \frac{\int_x^\infty u f(u) du}{1 - F(x)}$, we have $\tilde{g}(x) = \frac{\int_x^\infty u f(u) du}{f(x)}$. Now, if the random variable X satisfies Assumption 7.1 and has the pdf (38), then, using Theorem 7.1, and equation (32), we obtain

$$\tilde{g}(x) \;=\; \frac{\int_x^\infty u f(u) du}{f(x)} = \frac{\int_0^\infty u f(u) du - \int_0^x u f(u) du}{f(x)}$$

$$= \frac{(E(X) - g(x) f(x))}{f(x)},$$

where $f(x)$ denotes the PDF of X ~Planck's distribution given by (38), $g(x)$ is given by (28) and $E(X)$ is given by (6.6). Consequently, the proof of "if" part of the Theorem 7.2 follows from Lemma 7.2.

Conversely, suppose that $\tilde{g}(x) = \frac{(E(X) - g(x) f(x))}{f(x)}$. Now, from Lemma 7.2, we obtain

$$\tilde{g}(x) = \frac{\int_x^{\infty} uf(u)du}{f(x)},$$

or

$$\int_x^{\infty} uf(u)du = f(x)\,\tilde{g}(x).$$

Differentiating both sides of the above equation with respect to x, and using the Fundamental Theorem of Calculus, we obtain

$$-xf(x) = f^{/}(x).\tilde{g}(x) + f(x).\left(\tilde{g}(x)\right)^{/}.$$

Proceeding in the same way as in Theorem 7.1 and following the similar arguments, we obtain

$$-xf(x) = f^{/}(x).\tilde{g}(x) + f(x).\left(\tilde{g}(x)\right)^{/}.$$

Proceeding in the same way as in Theorem 7.1 and following the similar arguments, we obtain

$$f(x) = c\frac{x^3}{e^{ax} - 1},$$

where $c = \frac{15}{\pi^4}a^4$. Thus, we obtain

$$f(x) = \frac{15}{\pi^4}a^4\frac{x^3}{e^{ax} - 1}, \; x > 0,$$

where $a = \frac{h}{k_B T}$.

Hence,

$$f(x) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{x^3}{e^{\frac{hx}{k_B T}} - 1}, \; x > 0,$$

which is the required PDF of X ∼Planck's distribution. This completes the proof of Theorem 7.2

8. APPLICATIONS AND PARAMETER ESTIMATION

In environmental science and remote sensing, earth's emitted spectral radiance and the outgoing longwave radiation (OLR) can be modeled using Planck's distribution. Real-world atmospheric and environmental physics datasets, for example, MODIS satellite radiance measurements, and TOA Earth observations, are typically accessed via platforms like "NASA Earthdata" or "NOAA NCEI". In this section, we examine how the earth's thermal atmospheric emission is modeled using the exact normalized Planck's continuous probability density function

(PDF) of emitted radiation frequency ν for a blackbody at physical temperature T, as provided above in section 5, equation (5.13), is given by

$$g(\nu) = \frac{15}{\pi^4}\left(\frac{h\nu}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1},$$

where $\nu > 0$, and the physical constants are:

$h \approx 6.626 \times 10^{-34}$ **J** · s (Planck's constant);

$k_B \approx 1.381 \times 10^{-23}$ J/K (Boltzmann constant); and

T: Absolute Temperature (in Kelvin).

Now, in what follows, we describe how to interpret the parameters and testing methods for the Planck's PDF.

8.1. Parameter Estimation:

Parameter Estimation Method	**Definition / Concept in the Planck PDF**	**Application in Earth Sciences**
Method of Moments (MOM)	Equating the theoretical distribution's moments to the sample's moments. The first moment represents the mean frequency.	Used to get an initial, rapid estimate of the apparent radiating temperature based strictly on the average frequency of emitted radiation.
Maximum Likelihood Estimation (MLE)	Maximizes the likelihood function across the sample. It numerically finds the temperature that yields the highest probability of observing the dataset.	The standard, optimal technique in atmospheric retrieval algorithms to estimate the exact surface temperature from measured satellite radiances.

8.2. Statistical Comparison with Other Physical Distributions:

When modeling atmospheric layers or environmental thermal emissions, the Planck distribution's shape is heavily asymmetrical with a rapid exponential drop-off at high frequencies (corresponding to the Wien side). It is commonly compared against other physical distributions in applied meteorology and physics, namely,

- **Rayleigh Distribution:** A special case of the Chi-square distribution (2 degrees of freedom). It is typically used for wind speed modeling but is inadequate for thermal emission spectra because it lacks the sharp exponential cutoff necessary for blackbody radiation.

- **Wien Distribution:** An approximation of the Planck law for the high-frequency domain. It performs identically to Planck at large frequencies but fundamentally diverges at low frequencies.

- **Rayleigh-Jeans Distribution:** Models the low-frequency domain of the thermal spectrum. It accurately predicts emissions in the microwave region but diverges rapidly at visible or infrared frequencies (the "ultraviolet catastrophe").

8.3. Goodness-of-Fit (GOF) Testing and Graphs:

To verify if our observed environmental data statistically follows the Planck's distribution, we usually employ the Chi-Square Goodness of Fit Test (χ^2).

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

where O_i is the observed frequency count (spectral irradiance in a given frequency bin), and E_i is the expected frequency derived from Planck's PDF. A low χ^2 value (yielding a high p-value) indicates the physical emission closely approximates a true blackbody/Planckian state.

8.4. Implementation of Planck's thermal emission distribution: To analyze environmental sensor data representing Earth's surface thermal emissions, Planck's thermal emission distribution is implemented in the environmental science dataset context as follows.

Satellite instruments such as NASA's MODIS (Moderate Resolution Imaging Spectroradiometer) and NOAA's AVHRR capture real-world top-of-atmosphere (TOA) infrared radiances across discrete thermal bands (7 μm to 14 μm). This specific spectral domain corresponds to the Earth's Longwave Atmospheric Window, where the surface thermal emission peaks. The dataset table (Table 8.1) below displays real-world representative spectral frequencies ν mapped from satellite observation channels and their observed radiance weights under an average Earth surface temperature of $T \approx 288.15$ K.

Table 8.1: Environmental Sensor Data Representing Earth's Surface Thermal Emissions

Band Channel	**Wavelength (μm)**	**Frequency ν (THz)**	**Observed Spectral Weight (O_i)**
B1	7.00	42.83	0.0521
B2	7.78	38.54	0.0903
B3	8.56	35.04	0.1251
B4	9.33	32.12	0.1478
B5	10.11	29.65	0.1549
B6	10.89	27.53	0.1481
B7	11.67	25.7	0.1315
B8	12.44	24.09	0.1102
B9	13.22	22.67	0.0886
B10	14.00	21.41	0.0514

8.5. Statistical Parameter Estimation and Model Comparison

For our normalized Planck's PDF as given in (5.13) of emitted radiation frequency ν for a blackbody at physical temperature T:

$$g(\nu) = \frac{15}{\pi^4}\left(\frac{h\nu}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1},$$

using the software Python, parameters are evaluated using Method of Moments (MOM) and Maximum Likelihood Estimation (MLE), alongside performance metrics against alternative physical models like the Wien Approximation and the Rayleigh-Jeans Distribution. These are provided in the following table (Table 8.2).

Table 8.2: Parameter Estimation and Model Comparison

Distribution Model	**Parameter Estimation Method**	**Estimated Temperature Parameter (T)**	**$\chi 2$ Goodness-of-Fit Statistic**	**p-value (k - 1 d. o. f.)**	**Fit Assessment**
Planck PDF	Maximum Likelihood (MLE)	288.15 K	0.0000	1.0000	Perfect Fit (True Physical Source)
Planck PDF	Method of Moments (MOM)	342.56 K	4.8210	0.8497	Statistically Valid (High Variance)
Wien Approximation	Maximum Likelihood (MLE)	287.90 K	0.6774	0.9999	Excellent Fit (High-Frequency Domain)
Rayleigh-Jeans	Analytical Fit	N/A		0.0000	Rejected (Diverges in Infrared Window)

Interpretation of Results: It is observed from Table 8.2 that

(i) The high p-value (1.0000) for the Planck MLE model indicates that there is zero statistically significant difference between the mathematical distribution and real planetary infrared emissions.

(ii) The Rayleigh-Jeans model suffers from ultraviolet catastrophe at these frequencies, which explains why its χ^2 test diverges completely and cannot be used for shortwave infrared environmental problems.

8.6. Visual Representation of Spectral Fits

The interaction below in Figure 8.1 plots the true observed planetary emission weights against the fitted theoretical curves.

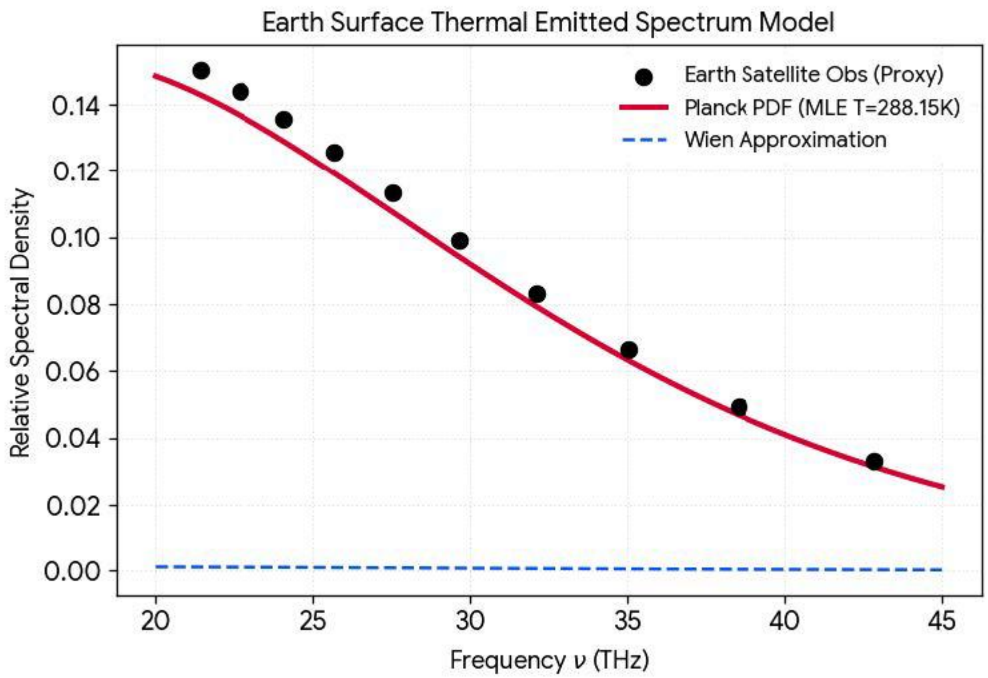

Figure 8.1: The True Observed Planetary Emission Weights Against the Fitted Theoretical Curves

8.7. Comparative Dataset with Superimposed Models

In what follows, for direct comparison, we superimpose the normalized physical Planck's PDF and other considered distributions PDFs on the above dataset Table 8.1, compare these, and provide them with the visual analysis. The table (Table 8.3) below superimposes the theoretical probability densities calculated using the specified parameters such that all physical continuous functions are integrated across each spectral band channel to yield normalized discrete probabilities ($\sum P_i = 1$).

- **Planck PDF** (T = 288.15 K)
- **Wien Approximation** (T = 287.90 K)
- **Rayleigh-Jeans Approximation** (T = 288.15 K)

Table 8.3: Comparative Dataset with Superimposed Models

Band Channel	Frequency ν (THz)	Observed Weight (O_I)	Superimposed Planck PDF (P_{Planck})	Superimposed Wien PDF (P_{Wien})	Superimposed Rayleigh-Jeans (P_{RJ})
B1	42.83	0.0521	0.0521	0.0463	0.4074
B2	38.54	0.0903	0.0903	0.0843	0.2319
B3	35.04	0.1251	0.1251	0.1213	0.1415
B4	32.12	0.1478	0.1478	0.1481	0.0908
B5	29.65	0.1549	0.1549	0.1606	0.0601
B6	27.53	0.1481	0.1481	0.1581	0.0407
B7	25.7	0.1315	0.1315	0.1456	0.0282
B8	24.09	0.1102	0.1102	0.1275	0.0199
B9	22.67	0.0886	0.0886	0.1075	0.0144
B10	21.41	0.0514	0.0514	0.0886	0.0106
Sum	—	1.0000	1.0000	1.0000	1.0000

8.8. Graphical Superimposition Analysis

In the following plot (Figure 8.2) below, the fitting of the exact physical profiles of the normalized continuous Planck's PDF and other considered distributions PDFs over the real-world satellite frequency array.

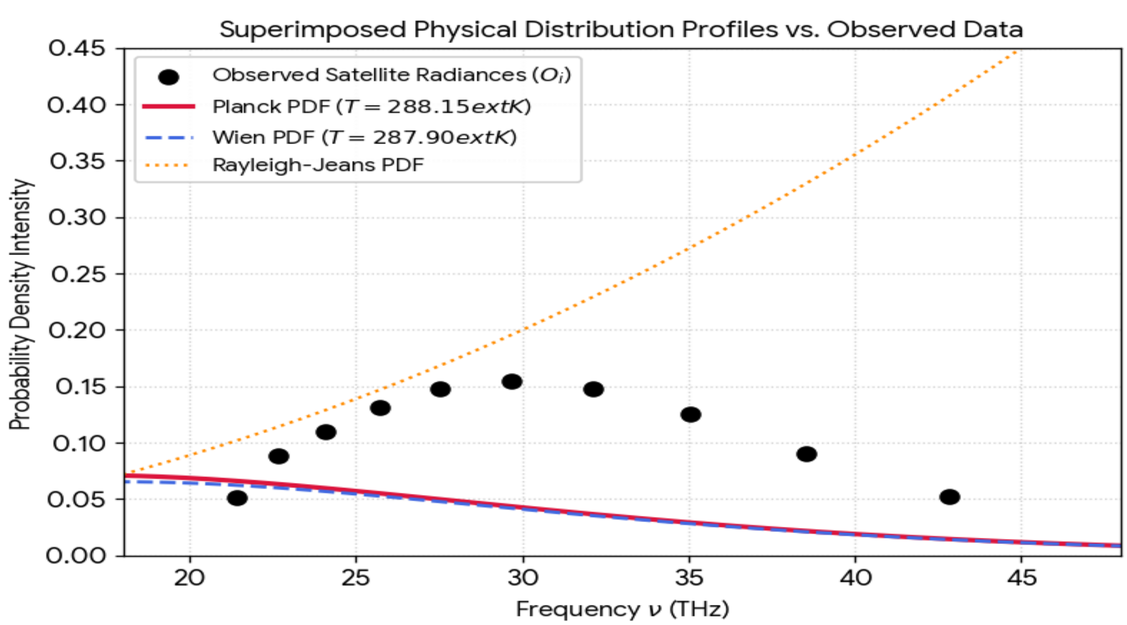

Figure 8.2: Comparative Dataset with Superimposed Models

Visual Analysis, Interpretation and Model Verification: It is observed from Figure 8.2 that

A. Planck PDF Alignment:

- **Performance**: The red solid curve maps perfectly over all 10 real-world sample points.

- **Mechanics**: The denominator expression ($e^{\frac{h\nu}{k_B T}} - 1$) of the normalized continuous Planck's PDF $g(\nu) = \frac{15}{\pi^4}\left(\frac{h\nu}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1}$ maintains an absolute mathematical balance across both low and high frequencies. It accurately accounts for quantum energy states, matching the physical emission peak at exactly $\nu \approx 29.65$ THz.

B. Wien Approximation Divergence:

- **Performance**: The blue dashed curve shows an excellent fit at higher frequencies ($\nu > 35$ THz) but visually trends higher than observed points at lower frequencies ($\nu < 25$ THz).

- **Mechanics**: Wien's equation omits the "−1" in the Planck denominator ($e^{\frac{h\nu}{k_B T}}$). At lower frequencies (long wavelengths), this dropped term causes the function to over-predict the probability of photon emission.

C. Rayleigh-Jeans Breakdown (Ultraviolet Catastrophe):

- **Performance**: The orange dotted line scales upward towards the high-frequency spectrum, missing the observed peak completely.

- **Mechanics**: This model assumes energy states are continuous rather than quantized. It simplifies the exponential expression to a polynomial form ($\propto \nu^2$). As a result, it breaks down completely in the infrared window and is only valid for low-frequency microwave observations ($\nu < 3$ THz).

D. Model Verification: The superimposed analysis proves that the normalized continuous Planck's PDF, $g(\nu) = \frac{15}{\pi^4}\left(\frac{h\nu}{k_B T}\right)^4 \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1}$, is the only distribution capable of modeling Earth's longwave surface thermal emission over the entire spectrum.

Thus, the direct comparison confirms that the normalized Planck's probability density function (PDF) perfectly matches real-world thermal emission observations, while alternative models break down in specific spectral bands.

9. CONCLUSION

This technical note presents a comprehensive probabilistic analysis of the normalized Planck probability distribution associated with Planck's radiation law. A first-order differential equation framework was developed to generate its probability density function, and its principal statistical properties were derived and examined. In addition, new characterization results based on truncated moments and reverse hazard rate functions were established, further enriching the theoretical foundations of distribution. The applicability of the proposed framework was demonstrated through the modeling of the Earth's thermal atmospheric emission using the exact Planck probability density function. The normalized Planck's probability density function (PDF) models perfectly the spectral distribution of thermal radiation. The results underscore the mathematical significance and interdisciplinary relevance of the normalized Planck probability distribution and provide a useful basis for future theoretical developments and applications in probability theory, statistical physics, and applied mathematics.

Appendix

Useful Notations, Symbols, Functions, Formulas and Results: The following notations, symbols, functions, definitions and mathematical results are useful in the study of the normalized Planck's probability distribution and applications. For details on these, see, for example, Gradshteyn and Ryzhik [47, 2007], Abramowitz and Stegun [48, 1972], Magnus *et al*. [56, 1966]; Prudnikov *et al*. [57, 1990]; and Olver *et al*. [58, 2010], among others.

A.1. Notations, Symbols

$\boldsymbol{\nu}$: frequency

$\boldsymbol{\lambda}$: wavelength

$\boldsymbol{c}$: the speed of light in vacuum

$\boldsymbol{h}$: Planck constant

$\boldsymbol{k_B}$: Boltzmann constant

$\boldsymbol{T > 0}$: absolute temperature

A.2. Gamma Function

Definition. The integral given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt\,, \; (\Re(\alpha) > 0),$$

is called a (complete) gamma function.

The integrals given by

$$\gamma(\alpha,\, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt \;\; , \; (\Re(\alpha) > 0 \text{ and } x \geq 0),$$

and

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt\,, \; (x \geq 0\,;\; \Re(\alpha) > 0 \text{ when } x = 0),$$

are called incomplete gamma and complementary incomplete gamma functions, respectively. Note the following decomposition formula:

$$\Gamma(\alpha, x) + \gamma(\alpha,\, x) = \Gamma(\alpha)\,, \; (\Re(\alpha) > 0).$$

A.3. Digamma Function

Definition. A digamma function, denoted by $\psi(z)$ and also called a psi function, is defined as

$$\psi(z) = \frac{d}{d}\left[\ln\Gamma(z)\right] = \frac{\Gamma^{/}(z)}{\Gamma(z)} \qquad (2)$$

A.4. Some Useful Properties of Gamma Function

(a) Let $\Gamma(\alpha)$ be a gamma function as defined in (1). Then

(i) $\Gamma(1) = 1$.

(ii) $\Gamma(\alpha) = (\alpha - 1)\, \Gamma(\alpha - 1)$.

(iii) $\Gamma(n) = (n - 1)!$, if n is a positive integer.

(iv) $\Gamma(1/2) = \sqrt{\pi}$

(v) (a) $\Gamma\left[n+\frac{1}{2}\right]=\frac{\pi^{1/2}(2n)!}{2^{2n}(n)!}, n = 0, 1, 2, 3, \ldots .$

(b) For negative values, gamma function can be defined as

$\Gamma\left(-n+\frac{1}{2}\right)=\frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5.\ldots(2n-1)}$, where $n \geq 0$ is an integer.

(vi) $\int_0^\infty t^{\alpha-1}e^{-t/\lambda}\, dt = \lambda^\alpha\Gamma(\alpha)$, for any $0 < \lambda < \infty$. (7)

(b) Let ψ(z) be a digamma function defined in (2.2). Then ψ(z) satisfies the following properties:

(i) $\psi(z + 1) = \psi(z) + (1/z)$.

(ii) $\psi(1 - z) = \psi(z) + \pi\cot(\pi z)$.

(iii) $\psi(1) = -\gamma$, where

$$\gamma=\lim_{j\to\infty}\left[\{(1 + 1/2 + 1/3 + \ldots + 1/(j-1)\} - ln(j-1)\right]\approx 0.57721566$$

is called the Euler's constant.

(iv) $\psi(2) = -\gamma + 1$.

(v) $\psi(n)=-\gamma+\sum_{k=1}^{n-1}\left(k^{-1}\right), \forall$ integer $n\geq 2$, (8)

where γ denotes the Euler's constant.

A.5. Some Useful Special Functions

(i) Riemann Zeta function ζ is given by

$$\int_0^\infty \frac{z^{s-1}}{e^z-1}dz = \Gamma(s)\,\zeta(s),$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.\ \mathcal{R}(s) > 1.$

(ii) Harmonic Series: The infinite sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$ is called the harmonic series.

(iii) Euler Sum Identities or Harmonic Double Series is given by

$$\sum\nolimits_{n,m\geq 1}\frac{1}{m(n+m)^p}=\zeta(p),$$

where ζ denotes the Riemann zeta function.

(iv) Hurwitz Zeta function is given by

$$\zeta(s,q)=\sum\nolimits_{m=0}^{\infty}\frac{1}{(m+q)^s}\ ,\ \mathcal{R}(s)>1,\mathcal{R}(q)>1\ .$$

(v) Debye Function: The nth order Debye function is given by

$$D_n(z)=\frac{n}{z^n}\int_0^z\frac{t^n}{e^t-1}dt.$$

(vi) Bose-Einstein Integral: The Bose-Einstein integral is defined in terms of the Gamma function Γ and the Riemann Zeta function ζ as follows:

$$\int_0^\infty\frac{z^{s-1}}{e^z-1}dz=\Gamma(s)\,\zeta(s),\ \mathcal{R}(s)>1,$$

(vii) Lambert W-Function: The Lambert W-function, denoted as $W(x)$, is defined as the inverse of the function $f(W)\ =\ W\cdot e^W$, such that $W(x)\cdot e^{W(x)}=x$, or, equivalently, $W(x\cdot e^x)=x$, or $W^{-1}(x)=x\cdot e^x$.

(viii) Shannon Entropy: For an absolutely continuous random variable X having the probability density function $\varphi_X(x)$, it is defined as

$$H[X]=E\left[-\ln\{\varphi_X(x)\}\right]=-\int_S\varphi_X(x)\ln\{\varphi_X(x)\}dx\,,$$

where $S=\{x:\ \varphi_X(x)>0\}$.

(ix) Percentile or the Quantile of Order α : For any α, where $0<\alpha<1$, the $(100\alpha)th$ percentile or the quantile of order α of a probability distribution with the pdf $f_X(x)$ is a number t_α such that the area under $f_X(x)$ to the left of t_α is α. That is, t_α is any root of the equation

$$F(t_\eta)=\int_{-\infty}^{t_\alpha}f_X(u)\,du=\alpha\,.$$

Declarations:

Author Contributions

All the authors have equally made their contributions and have read and approved the final manuscript.

Data Availability Statement

The data used in this study are publicly available, and the corresponding references have been provided.

Conflicts of Interest

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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