PRICING TEMPERATURE DERIVATIVES UNDER A TIME-CHANGED LEVY MODEL

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Abstract. The objective of the paper is to price weather derivative contracts using temperature as the underlying process when the later follows a mean-reverting dynamics driven by a time-changed Brownian motion coupled to a Gamma Levy subordinator and time-dependent deterministic volatility. This type of model captures the complexity of the temperature dynamic providing a more accurate valuation of their associated weather contracts. An approximated price is obtained by a Fourier expansion of its characteristic function combined with a selection of the equivalent martingale measure following the Esscher transform as proposed in Gerber and Shiu (1994).

1. Introduction

The objective of the paper is to price weather derivative contracts using temperature as the underlying process when the later follows a mean-reverting dynamics driven by a time-changed Brownian motion coupled to a Gamma Levy subordinator and a time-dependent volatility function. The process reverts to a seasonal periodic deterministic process, while the volatility is considered also a periodic function of time, see Dacunha-Castelle, Hoang and Parey (2015) for the later. Temperature models driven by Levy noises and stochastic volatility have been originally considered in Benth and Benth-S(2009). This type of model captures some aspects of the complexity in the temperature dynamic, providing a more accurate valuation of the corresponding weather contracts. We follow an arithmetic version of the model in Switshchuk and Cui(2013), applied to the analysis of temperatures in the wetland of Everglades, Florida US. Our approach differs from the former in the pricing method, payoff and data location. Moreover, we use a Gamma subordinator instead of a Normal Inverse Gaussian one.
The availability of an explicit analytical expression of the characteristic function of the process allows for its Fourier expansion, which in turn leads to compute the approximated price under an equivalent martingale measure(EMM) obtained after an Esscher transform. See Gerber and Shiu (1994) for a rationale in terms of a utility-maximization criteria.
Methods based on Fourier expansions of the characteristic function in one

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and two dimensions have been implemented in Fang and Oosterlee (2008) to European contracts and further extended to other derivatives by the same authors, see Fang and Oosterlee (2014).
The combination of these three elements, namely the model, the pricing method and the choice of the EMM in the context of weather derivatives offers an efficient methodology for pricing such contracts.
The organization of the paper is the following:
In section 2 we describe the main model for the temperature process and obtain the characteristic function associated with it on both the historic and risk neutral dynamic. In section 3 we discuss the implementation of the Fourier expansion techniques, while in section 4 we show the numerical results regarding fitting of the model to the data, characteristic function, Gerber-Shiu parameter and Monte Carlo simulation.

2. A mean-reverting model temperature with Levy noises

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space verifying the usual conditions. For a stochastic process $(X_t)_{t\geq 0}$ defined on the space filtered space above the functions φ_{X_t} and $l_{X_t}(u) = \frac{1}{t}\log \varphi_{X_t}(-iu)$ represent its characteristic function and the cumulant generating function respectively. When the process has stationary and independent increments the later does not depend on t. The discounted process $(\tilde{X}_t)_{t\geq 0}$ is defined as $\tilde{X}_t = e^{-rt}X_t$, where r is the constant interest rate.
Let $(T_t)_{t\geq 0}$ be the temperature process defined on the filtered space above. We assumed it verifies the mean-reverting stochastic differential equation:

$$dT_t = ds_t + \alpha(s_t - T_t)dt + \sigma_t dV_t \tag{1}$$

where the temperature reverts to the deterministic seasonal process $(s_t)_{t\geq 0}$ given by:

$$s_t = \beta_0 + \beta_1 t + \beta_2 \sin\left(\frac{2\pi}{365}t\right) + \beta_3 \cos\left(\frac{2\pi}{365}t\right) \tag{2}$$

The parameter $\alpha > 0$ is the mean-reversion rate.
The deterministic volatility process $(\sigma_t)_{t\geq 0}$ and background noise $(V_t)_{t\geq 0}$ will be specified later on.
Changes of the average temperature over an interval $[t, t+h)$ are denoted $\Delta T_t = T_{t+h} - T_t$. In particular when $h = \frac{1}{365}$ daily changes in temperature are considered, measured in year units. Notice that the data consist in temperature daily averages, whereas in pricing derivative contracts time is usually measured in years.
The solution of equation (1) is given in the following elementary lemma.

Lemma 1. *The solution of equation (1) is:*

$$T_t = D_1(t, \alpha) + e^{-\alpha t}W_t \tag{3}$$

with $W_t = \int_0^t \sigma_u e^{\alpha u} dV_u$ *and* $D_1(t, \alpha) = s_t + e^{-\alpha t}(T_0 - s_0)$

Proof. We apply Ito formula to the function $f(x,y) = xe^{\alpha y}$ and the process $(T_t, t)_{t\geq 0}$. Hence:

$$\begin{aligned}
T_t e^{\alpha t} &= T_0 + \int_0^t e^{\alpha u} dT_u + \alpha \int_0^t e^{\alpha u} T_{u^-} du \\
&+ \sum_{u\leq t} [T_u e^{\alpha u} - T_{u^-} e^{\alpha u} - \Delta T_{u^-} e^{\alpha u}] \\
&= T_0 + \int_0^t e^{\alpha u} ds_u + \alpha \int_0^t e^{\alpha u} s_u du - \alpha \int_0^t e^{\alpha u} T_{u^-} du \\
&+ \int_0^t e^{\alpha u} \sigma_u dV_u + \alpha \int_0^t e^{\alpha u} T_{u^-} du \\
&= T_0 + \int_0^t e^{\alpha u} s'_u du + \alpha \int_0^t e^{\alpha u} s_u du + \int_0^t e^{\alpha u} \sigma_u dV_u
\end{aligned}$$

Multiplying by $e^{-\alpha t}$ on both sides leads to equation (3). □

We further assume that the volatility also follows a deterministic seasonal component process:

$$\sigma_t = c_0 + c_1 t + c_2 \sin\left(\frac{2\pi}{365} t\right) + c_3 \cos\left(\frac{2\pi}{365} t\right) \tag{4}$$

where $c_j \geq 0, j = 0, 1, 2, 3$.
We need to compute the characteristic function of an integral of the background noise process given by W_t. To this end we refer to a well-known result about the functional of a Levy process $(\xi_t)_{t\geq 0}$ with $\xi_0 = 0$ and a measurable function f given by:

$$E(exp(i \int_0^t f(s)\, d\xi_s)) = exp(\int_0^t l_\xi(-if(s))\, ds) \tag{5}$$

See for example Eberlein and Raible (1999).
Specifically, for a stochastic process $(X_t)_{t\geq 0}$ we consider its Esscher transform:

$$\frac{d\mathcal{Q}_t^\theta}{dP_t} = \exp(\theta X_t - t l_X(\theta)),\ 0 \leq t \leq T,\ \theta \in \mathbb{R} \tag{6}$$

where P_t and $\mathcal{Q}_t^\theta$ are the respective restrictions of P and $\mathcal{Q}^\theta$ to the σ-algebra $\mathcal{F}_t$. By $\varphi_{X_t}^\theta$ and $l_X^\theta(u)$ are defined respectively the characteristic function and cumulant generating function of a process $(X_t)_{t\geq 0}$ under the probability $\mathcal{Q}^\theta$ obtained by an Esscher transformation as given in equation (6). The value $\theta \in \mathbb{R}$ is a parameter in the Esscher transform.
For consistency we denote $\varphi_{X_t}^0 := \varphi_{X_t}$ and $l_X^0 = l_X$.
The expected value under $\mathcal{Q}^\theta$ is denoted by E_θ.
A subordinator process $(R_t)_{t\geq 0}$ and a time-changed process $(V_t)_{t\geq 0}$ are introduced in a way that they verify:

$$V_t = B_{R_t} + \mu_1 R_t \tag{7}$$

Here $\mu_1 \in \mathbb{R}$ is a parameter in the model and $(B_t)_{t\geq 0}$ is a standard Brownian motion.
The following result describes the characteristic function of the temperature process under the historic and EMM measures.

Proposition 2. *For the model given by equations (1), (2) and (7) the characteristic function of T_t under the probability P is:*

$$\varphi_{T_t}(u) \quad = \quad C_1(t,u,\alpha)\exp(I(0,t,u)) \tag{8}$$

where:

$$\begin{aligned} C_1(t,u,\alpha) &= \exp(iu(s_t - e^{-\alpha t}s_0 + e^{-\alpha t}T_0)) = \exp(iuD_1(t,\alpha)) \\ v_1(t,s,u) &= u\sigma_s e^{-\alpha t} \end{aligned}$$

Moreover, under the EMM $\mathcal{Q}^\theta$:

$$\varphi^{\theta}_{T_t}(u) \quad = \quad C_1(t,u,\alpha)C_2(t,\theta)\exp(I^\theta(0,t,u)) \tag{9}$$

where:

$$\begin{aligned} C_2(t,\theta) &= \exp(-tl_R(\theta\mu_1 + \frac{1}{2}\theta^2)) \\ I^\theta(a,t,u) &= \int_a^t l_R(-i\mu_1 v_1(t,s,u)e^{\alpha s} + \mu_1\theta + \frac{1}{2}(-iv_1(t,s,u)e^{\alpha s} + \theta)^2)ds \end{aligned}$$

and $I(a,t,u) = I^0(a,t,u)$.

Proof. By conditioning:

$$\begin{aligned} \varphi_{V_t}(u) &= E[E[\exp(i(uV_t)/R_t)]] = E[\exp(i\mu_1 uR_t)E[\exp(iuB_{R_t}/R_t)]] \\ &= E[\exp(i\mu_1 uR_t)\exp(-\frac{1}{2}R_t u^2)] = E[\exp(i(\mu_1 u + \frac{1}{2}iu^2)R_t)] \\ &= \varphi_{R_t}(\mu_1 u + \frac{1}{2}iu^2) \end{aligned}$$

Hence:

$$l_V(u) = l_R(\mu_1 u + \frac{1}{2}u^2) \tag{10}$$

By Lemma 1:

$$\begin{aligned} \varphi_{T_t}(u) &= E[e^{iuT_t}] = C_1(t,u,\alpha)E[exp(iu\int_0^t \sigma_s e^{-\alpha(t-s)}dV_s)] \\ &= C_1(t,u,\alpha)\exp(\int_0^t l_V(-iu\sigma_s e^{-\alpha(t-s)})ds) \end{aligned} \tag{11}$$

The previous result combined with equations (10)and (5) leads to equation (8).
For the second part, notice that:

$$\varphi^{\theta}_{V_t}(u) \quad = \quad E(e^{iuV_t}e^{\theta V_t - tl_V(\theta)}) = \frac{\varphi_{V_t}(u-i\theta)}{\varphi_{V_t}(-i\theta)}$$

and $l_V^\theta(u) = l_V(u+\theta) - l_V(\theta)$.
Noticing that:

$$\begin{aligned}
\varphi_{T_t}^\theta(u) &= C_1(t,u,\alpha)\exp\left(\int_0^t l_V^\theta(-iu\sigma_s e^{-\alpha(t-s)})ds\right) \\
&= C_1(t,u,\alpha)\exp(-tl_V(\theta))\exp\left(\int_0^t l_V(-iu\sigma_s e^{-\alpha(t-s)}+\theta)ds\right)
\end{aligned}$$

from which equation (9) follows.

□

The result below specifies the value of θ under the Esscher transform.

Proposition 3. *Let $(T_t)_{t\geq 0}$ be the temperature process defined by equations (1), (2) and (7). Then, the Esscher measure $\mathcal{Q}^\theta$ is an EMM if for any $T>0$ the parameter θ verifies:*

$$l_V'(\theta) = -e^{(\alpha+r)T}(T_0 - \tilde{D}_1(T,\alpha))K_2^{-1}(\alpha,T) \tag{12}$$

where:

$$\begin{aligned}
K_2(\alpha,T) &= \int_0^T \sigma_u e^{\alpha u}\, du = \frac{c_0}{\alpha}(e^{\alpha T}-1) + \frac{c_1 T}{\alpha}e^{\alpha T} - \frac{c_1}{\alpha^2}(e^{\alpha T}-1) \\
&- \frac{365}{2\pi}c_2\left(\cos\left(\frac{2\pi}{365T}\right)-1\right) + \frac{365}{2\pi}c_3\left(\sin\left(\frac{2\pi}{365T}\right)-1\right)
\end{aligned}$$

Proof. From Lemma 1 the discounted temperature process $(\tilde{T}_t)_{t\geq 0}$ verifies:

$$\tilde{T}_t = \tilde{D}_1(t,\alpha) + e^{-\alpha t}\tilde{W}_t$$

It is a $\mathcal{Q}^\theta$-martingale if and only if for any $0 \leq s < t$:

$$\begin{aligned}
E_\theta(\tilde{T}_t/\mathcal{F}_s) &= \tilde{T}_s \\
\Leftrightarrow E_\theta(e^{-\alpha t}\tilde{W}_t - e^{-\alpha s}\tilde{W}_s/\mathcal{F}_s) &= \tilde{D}_1(s,\alpha) - \tilde{D}_1(t,\alpha)
\end{aligned}$$

But:

$$\begin{aligned}
E_\theta(e^{-\alpha t}\tilde{W}_t - e^{-\alpha s}\tilde{W}_s/\mathcal{F}_s) &= E_\theta\left(e^{-(\alpha+r)t}\int_0^t \sigma_u e^{\alpha u}\, dV_u - e^{-(\alpha+r)s}\int_0^s \sigma_u e^{\alpha u}\, dV_u/\mathcal{F}_s\right) \\
&= E_\theta\left(e^{-(\alpha+r)t}\int_s^t \sigma_u e^{\alpha u}\, dV_u + (e^{-(\alpha+r)t} - e^{-(\alpha+r)s})\int_0^s \sigma_u e^{\alpha u}\, dV_u/\mathcal{F}_s\right) \\
&= E_\theta\left(e^{-(\alpha+r)t}\int_s^t \sigma_u e^{\alpha u}\, dV_u\right) \\
&+ (e^{-(\alpha+r)t} - e^{-(\alpha+r)s})\int_0^s \sigma_u e^{\alpha u}\, dV_u
\end{aligned}$$

On the other hand, from equation (5):

$$\varphi^\theta_{\int_s^t \sigma_u dV_u}(x) = \exp\left(\int_s^t l_V^\theta(-ix\sigma_u e^{\alpha u})\, du\right) = exp\left(\int_s^t (l_V(-ix\sigma_u e^{\alpha u}+\theta) - l_V(\theta))\, du\right) \tag{13}$$

Hence:

$$
\begin{aligned}
E_\theta[e^{-(\alpha+r)t}\int_s^t \sigma_u e^{\alpha u}\, dV_u] &= e^{-(\alpha+r)t}\frac{1}{i}(\varphi^\theta_{\int_s^t \sigma_u e^{\alpha u}\, dV_u})'(x)|_{x=0} \\
&= -e^{-(\alpha+r)t}\left[\frac{1}{i}(i\int_s^t \sigma_u e^{\alpha u} l_V'(-ix\sigma_u e^{\alpha u}+\theta)\, du|_{x=0})\right. \\
&\qquad \left.\exp(\int_s^t (l_V(-ix\sigma_u e^{\alpha u}+\theta)-l_V(\theta))\, du|_{x=0})\right] \\
&= -e^{-(\alpha+r)t} l_V'(\theta)\int_s^t \sigma_u e^{\alpha u}\, du
\end{aligned}
$$

In particular, for $t = T$ and $s = 0$ the result in equation (12) follows from elementary calculation. □

Example 4. *A model with Gamma subordinator*
Consider the subordinator $(R_t)_{t\geq 0}$ is a Gamma process with parameters $a > 0, b > 0$, see Carr and Madan (1999). The respective characteristic function and Laplace exponent are:

$$
\begin{aligned}
\varphi_{R_t}(u) &= \left(1-\frac{iu}{b}\right)^{-at},\ a>0, b>0 \\
l_R(u) &= -a\log\left(1-\frac{u}{b}\right),\ u<b
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\varphi_{V_t}(u) &= \varphi_{R_t}(\mu_1 u + \frac{1}{2}iu^2) = \left(1-\frac{i\mu_1 u}{b}+\frac{1}{2b}u^2\right)^{-at} \\
l_V(u) &= -a\log[A_1(u)]
\end{aligned}
$$

where:

$$
A_1(u) = 1-\frac{\mu_1 u}{b}-\frac{1}{2b}u^2
$$

Moreover:

$$
\begin{aligned}
l_V^\theta(u) &= l_V(u+\theta)-l_V(\theta) \\
&= -a\left[\log A_1(u+\theta)-\log A_1(\theta)\right] \\
&= -a\log\left(\frac{A_1(u+\theta)}{A_1(\theta)}\right)
\end{aligned}
$$

Then, in this case we have:

$$
\begin{aligned}
C_2(t,\theta) &= \exp(-tl_R(\mu_1\theta+\frac{1}{2}\theta^2)) = A_1^{at}(\theta) \\
I^\theta(0,t,u) &= \exp(\int_0^t l_V^\theta(-iu\sigma_s e^{-\alpha(t-s)})ds) \\
&= \exp\left(-a\int_0^t \log\left(\frac{A_1(-iu\sigma_s e^{-\alpha(t-s)}+\theta)}{A_1(\theta)}\right)\, ds\right)
\end{aligned}
$$

The integral in the expression above is calculated numerically.
We compute the Gerber-Shiu parameter θ from the martingale condition given by equation (12). Namely:

$$e^{-(\alpha+r)T}K_2(\alpha,T)l'_V(\theta) = (T_0 - e^{-rT}D_1(T,\alpha))$$

Since

$$l'_V(\theta) = \frac{a(\mu_1+\theta)}{bA_1(\theta)}$$

the value θ that solves:

$$\frac{b}{a}e^{(\alpha+r)T}[T_0 - e^{-rT}D_1(T,\alpha)]K_2^{-1}(\alpha,T)A_1(\theta) - \theta - \mu_1 = 0 \tag{14}$$

makes the discounted prices martingales under the Esscher transformation.

3. Pricing temperature contracts

Weather contracts are based on cumulate temperatures (CAT), heating-degrees-days (HDD) or cooling-degrees-days (CDD) over certain period $[0,T]$ containing n days. Futures and option contracts are offered in Chicago Mercantile Exchange. They are respectively defined as:

$$\begin{aligned} \xi_T &= CAT = \sum_{k=1}^{n} T_k \\ \xi_{2,T} &= HDD = \sum_{k=1}^{n}(c - T_k)_+ \\ \xi_{3,T} &= CDD = \sum_{k=1}^{n}(T_k - c)_+ \end{aligned} \tag{15}$$

The typical value is $c = 65^0$ Fahrenheit degrees or 18^0 Celsius.
For concreteness we focus on a CAT index. To this end, for convenience, we rewrite the CAT index as:

$$\xi_T = \sum_{k=1}^{n}(T_0 + \sum_{j=1}^{k}\Delta T_j) = nT_0 + \sum_{j=1}^{n}\gamma_j \Delta T_j \tag{16}$$

where daily changes in the temperature are considered, i.e. $h = \frac{1}{365}$. They are independent random variables because of the underling Levy process $(V_t)_{t\geq 0}$. Here $\gamma_j = n - j + 1$.
A general payoff of the temperature weather derivative, consisting in a combination of a European long put and a long call with different strikes, known as *strangle*, is given by:

$$h(\xi_T) = d_1(\xi_T - K_1)_+ + d_2(K_2 - \xi_T)_+, d_j > 0, K_1 > K_2 > 0 \ , j = 1,2 \tag{17}$$

where d_1 and d_2 are the costs per unit of temperature below (resp. above) the threshold K_1 (resp. K_2) known as *tick sizes*. Typically $d_1 = d_2 = \$\ 20$. The price of a temperature contract over the period $[0, T]$ is :

$$
\begin{aligned}
p &= d_1 e^{-rT} E_{\mathcal{Q}}(\xi_T - K_1)_+ + d_2 e^{-rT} E_{\mathcal{Q}}(K_2 - \xi_T)_+ \\
&= d_1 e^{-rT} I_1 + d_2 e^{-rT} I_2
\end{aligned}
\tag{18}
$$

where $f_{\xi_T}(x, \theta)$ is the probability density function (p.d.f.) of the cumulated temperature index under the EMM measure and

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}} (x - K_1)_+ f_{\xi_T}(x, \theta)\ dx \\
I_2 &= \int_{\mathbb{R}} (K_2 - x)_+ f_{\xi_T}(x, \theta)\ dx
\end{aligned}
$$

In the same lines than in proposition 2 the characteristic function for the increments of the process can be computed. Let's denote the increments at points Δj by $\Delta X_j = X_{\Delta j} - X_{\Delta(j-1)}$ and $I_1^\theta(j-1, j, u) := I_1^\theta(\Delta(j-1), \Delta j, u)$, etc. Then, we have:

Theorem 5.

$$
\begin{aligned}
\varphi^\theta_{\Delta T_j}(u) &= L_1(j, u, \alpha) C_1(j-1, -u(1 - e^{-\alpha\Delta}), \alpha) C_2(j, \theta) \\
&\quad \exp[I^\theta(0, j-1, -u(1 - e^{-\alpha\Delta}))] \exp[I^\theta(j-1, j, u)]
\end{aligned}
\tag{19}
$$

$$
\varphi^\theta_{\xi_T}(u) = exp[L_1(\theta) + iuL_2(u, \alpha) + L_3(u, \alpha) + +L_4(u, \alpha)]
\tag{20}
$$

where:

$$
\begin{aligned}
L_1(j, u, \alpha) &= \exp[iu(D_1(j, \alpha) - e^{-\alpha\Delta} D_1(j-1, \alpha))] \\
v_1(j, u, s) &= u\sigma_s e^{-\alpha\Delta j} \\
I^\theta(j-1, j, u) &= \int_{\Delta(j-1)}^{\Delta j} l_R(-i\mu_1 v_1(j, u, s) e^{\alpha s} + \mu_1\theta + \frac{1}{2}(-iv_1(j, u, s) e^{\alpha s} + \theta)^2) ds \\
I^\theta(0, j-1, -u(1 - e^{-\alpha\Delta})) &= \int_0^{\Delta(j-1)} l_R(-i\mu_1 v_1(j, -u(1 - e^{-\alpha\Delta}), s) e^{\alpha s} + \mu_1\theta \\
&+ \frac{1}{2}(-iv_1(j, -u(1 - e^{-\alpha\Delta}), s) e^{\alpha s} + \theta)^2) ds
\end{aligned}
$$

and

$$\begin{aligned}
L_1(\theta) &= iunT_0 - \Delta\frac{n(n+1)}{2}l_R(\mu_1\theta + \frac{1}{2}\theta^2) \\
L_2(u,\alpha) &= -\sum_{j=1}^{n}\gamma_j(D_1(j,\alpha) - D_1(j-1,\alpha)) \\
L_3(u,\alpha) &= \sum_{j=1}^{n} I^{\theta}(0, j-1, -\gamma_j u(1-e^{-\alpha\Delta})) \\
L_4(u,\alpha) &= \sum_{j=1}^{n} I^{\theta}(j-1, j, \gamma_j u)
\end{aligned}$$

Proof. Let's denote $W_j = W_{\Delta j}$. From equation (3):

$$\begin{aligned}
\Delta T_j &:= T_{\Delta j} - T_{\Delta(j-1)} = D_1(j,\alpha) - D_1(j-1,\alpha) + e^{-\alpha\Delta j}W_j - e^{-\alpha\Delta(j-1)}W_{j-1} \\
&= D_1(j,\alpha) - D_1(j-1,\alpha) + e^{-\alpha\Delta j}\int_0^{\Delta j}\sigma_s e^{\alpha s}dV_s - e^{-\alpha\Delta(j-1)}\int_0^{\Delta(j-1)}\sigma_s e^{\alpha s}dV_s \\
&= D_1(j,\alpha) - D_1(j-1,\alpha) - e^{-\alpha\Delta(j-1)}(1-e^{-\alpha\Delta})\int_0^{\Delta(j-1)}\sigma_s e^{\alpha s}dV_s + e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s
\end{aligned}$$

Re-arranging the terms:

$$\begin{aligned}
\Delta T_j &= -(D_1(j-1,\alpha) + (1-e^{-\alpha\Delta})e^{-\alpha\Delta(j-1)}W_{j-1}) + D_1(j,\alpha) + e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s \\
&= -(1-e^{-\alpha\Delta})[(1-e^{-\alpha\Delta})^{-1}D_1(j-1,\alpha) + e^{-\alpha\Delta(j-1)}W_{j-1}] + D_1(j,\alpha) \\
&+ e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s \\
&= -(1-e^{-\alpha\Delta})[(1-e^{-\alpha\Delta})^{-1}D_1(j-1,\alpha) - D_1(j-1,\alpha) + D_1(j-1,\alpha) + e^{-\alpha\Delta(j-1)}W_{j-1}] \\
&+ D_1(j,\alpha) + e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s
\end{aligned}$$

Hence:

$$\begin{aligned}
\Delta T_j &= -(1-e^{-\alpha\Delta})[(1-e^{-\alpha\Delta})^{-1}D_1(j-1,\alpha) - D_1(j-1,\alpha) + T_{j-1}] \\
&+ D_1(j,\alpha) + e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s \\
&= -e^{-\alpha\Delta}D_1(j-1,\alpha) - (1-e^{-\alpha\Delta})T_{j-1} + D_1(j,\alpha) + e^{-\alpha\Delta j}\int_{\Delta(j-1)}^{\Delta j}\sigma_s e^{\alpha s}dV_s
\end{aligned}$$

Notice that the temperature increments splits into two independent terms plus a constant. Therefore, combining equations (9) and again equation (5) we have, under the selected EMM and taking into account equation (9) that:

$$\begin{aligned}
\varphi^{\theta}_{\Delta T_j}(u) &= L_1(j,u,\alpha)\varphi^{\theta}_{T_{j-1}}(-u(1-e^{-\alpha\Delta}))\exp(\int_{\Delta(j-1)}^{\Delta j} l^{\theta}_V(-iv_1(j,u,s)e^{\alpha s})ds) \\
&= L_1(j,u,\alpha)C_1(j-1,-u(1-e^{-\alpha\Delta}),\alpha)C_2(j-1,\theta)\exp[I^{\theta}(0,j-1,-u(1-e^{-\alpha\Delta}))] \\
&\quad e^{-\Delta l_V(\theta)}\exp(\int_{\Delta(j-1)}^{\Delta j} l_V(-iv_1(j,u,s)e^{\alpha s}+\theta)ds) \\
&= L_1(j,u,\alpha)C_1(j-1,-u(1-e^{-\alpha\Delta}),\alpha)C_2(j-1,\theta)\exp[I^{\theta}(0,j-1,-u(1-e^{-\alpha\Delta}))] \\
&\quad e^{-\Delta l_R(\mu_1\theta+\frac{1}{2}\theta^2)}\exp(I^{\theta}_1(j-1,j,u))
\end{aligned}$$

It easily leads to equation (19).
Furthermore, from equation (16) and the result above:

$$\varphi^{\theta}_{\xi_T}(u) = \varphi^{\theta}_{nT_0+\sum_{j=1}^{n}\gamma_j\Delta T_j}(u) = E_{\theta}[e^{iu(nT_0+\sum_{j=1}^{n}\gamma_j\Delta T_j)}] = e^{iunT_0}\prod_{j=1}^{n}\varphi^{\theta}_{\Delta T_j}(\gamma_j u)$$

Replacing expression (19) into the last equation we have

$$\begin{aligned}
\varphi^{\theta}_{\xi_T}(u) &= e^{iunT_0}\prod_{j=1}^{n}\varphi^{\theta}_{\Delta T_j}(\gamma_j u) \\
&= e^{iunT_0}\prod_{j=1}^{n}L_1(j,\gamma_j u,\alpha)\prod_{j=1}^{n}C_1(j-1,-\gamma_j u(1-e^{-\alpha\Delta}),\alpha)\prod_{j=1}^{n}C_2(j,\theta) \\
&\quad \prod_{j=1}^{n}\exp[I^{\theta}_1(0,j-1,-\gamma_j u(1-e^{-\alpha\Delta}))]\prod_{j=1}^{n}\exp(I^{\theta}_1(j-1,j,\gamma_j u)) \\
&= e^{iunT_0}\exp[iu(\sum_{j=1}^{n}\gamma_j(D_1(j,\alpha)-e^{-\alpha\Delta}D_1(j-1,\alpha)))]\exp(-iu\Delta l_R(\theta\mu_1+\frac{1}{2}\theta^2)\sum_{j=1}^{n}j) \\
&\quad \exp[-iu(1-e^{-\alpha\Delta})\sum_{j=1}^{n}\gamma_j D_1(j-1,\alpha)]\exp[\sum_{j=1}^{n}I^{\theta}(0,j-1,-\gamma_j u(1-e^{-\alpha\Delta}))] \\
&\quad \exp[\sum_{j=1}^{n}I^{\theta}(j-1,j,\gamma_j u)]
\end{aligned}$$

From which equation (20) follows. □

3.1. **Pricing by cosine Fourier expansion.** A Fourier expansion of the p.d.f. $f_{\xi_T}(x,\theta)$ on an interval $[b_1,b_2]$ is given by:

$$f_{\xi_n}(x,\theta) = \sum_{k=0}^{+\infty}A_k(\theta)cos\left(k\pi\frac{x-b_1}{b_2-b_1}\right) \tag{21}$$

where the coefficients in the expansion, the first of them divided by two, are:

$$A_k(\theta) \quad = \quad \frac{2}{b_2 - b_1} exp\left(-i\frac{k\pi b_1}{b_2 - b_1}\right)\varphi^{\theta}_{\xi_T}\left(\frac{k\pi}{b_2 - b_1}\right)$$

(22)

for b_1 and b_2 large enough.
It leads to the approximate calculations:

$$I_1 \quad \simeq \quad \frac{b_5^2 - b_4^2}{2(b_5 - b_3)} + 2\sum_{k=1}^{N_1} exp\left(-i\frac{k\pi b_1}{b_5 - b_3}\right)\varphi^{\theta}_{\xi_T}\left(\frac{k\pi}{b_5 - b_3}\right)R_1(k)$$

and

$$I_2 \quad \simeq \quad \frac{b_7^2 - b_6^2}{2(b_8 - b_7)} + 2\sum_{k=1}^{N_1} exp\left(-i\frac{k\pi b_1}{b_8 - b_7}\right)\varphi^{\theta}_{\xi_T}\left(\frac{k\pi}{b_8 - b_7}\right)R_2(k)$$

where $b_3 = b_1 - K_1$, $b_4 = (b_1 - K_1, 0)_+$, $b_5 = b_2 - K_1$, $b_6 = (K_2 - b_2)_+$, $b_7 = K_2 - b_2$ and $b_8 = K_2 - b_1$.
Here:

$$\begin{aligned} R_1(k) &= -\frac{b_4}{k\pi}\sin\left(k\pi\frac{b_4 - b_3}{b_5 - b_3}\right) + \frac{b_5 - b_3}{k^2\pi^2}\left((-1)^k - \cos\left(k\pi\frac{b_4 - b_3}{b_5 - b_3}\right)\right) \\ R_2(k) &= -\frac{b_6}{k\pi}\sin\left(k\pi\frac{b_6 - b_8}{b_8 - b_7}\right) + \frac{b_8 - b_7}{k^2\pi^2}\left((-1)^k - \cos\left(k\pi\frac{b_6 - b_8}{b_8 - b_7}\right)\right) \end{aligned}$$

Details in the calculations, which are adapted from Fang and Oosterlee (2008) to this particular case are rather straightforward, have been left to the appendix.
The delicate choice of the truncation values b_1 and b_2 as well as the number of terms in the truncated expansion depends on the model considered. For a discussion of the error analysis in the truncation and numerical errors present in the Fourier Cosine method in the case of the exponential Levy model class we refer the reader to the work of Fang and Oosterlee (2008).

3.2. **Pricing by inverse Fourier transform.** Assume there exist real values $R_1 > 1$ and $R_2 < -1$ such that $E_{\mathcal{Q}}[e^{RV_t}] < +\infty$. The values $R_j, j = 1, 2$ are damping factors to account for integrability. See Raible(2000).
Let's denote the payoff functions by $h_1(x) = (x - K_1)_+$ and $h_2(x) = (K_2 - x)_+$ and $h_{R,j}(x) = e^{-R_j x}h_j(x) \in L^1(\mathbb{R}), j = 1, 2$. Its Fourier transform is $\hat{h}_j(x)$.

Hence:

$$\begin{aligned}
p_W &= d_1 e^{-rT} E_{\mathcal{Q}}(\xi_T - K_1)_+ + d_2 e^{-rT} E_{\mathcal{Q}}(K_2 - \xi_T)_+ \\
&= d_1 e^{-rT} \int_{\mathbb{R}} e^{R_1 y} h_{R,1}(y) f_{\xi_T}(y,\theta)\, dy + d_2 e^{-rT} \int_{\mathbb{R}} e^{R_2 y} h_{R,2}(y) f_{\xi_T}(y,\theta)\, dy \\
&= \frac{1}{2\pi} d_1 e^{-rT} \int_{\mathbb{R}} e^{R_1 y} \left[\int_{\mathbb{R}} e^{-ixy} \hat{h}_{R,1}(x)\, dx\right] f_{\xi_T}(y,\theta)\, dy \\
&+ \frac{1}{2\pi} d_2 e^{-rT} \int_{\mathbb{R}} e^{R_2 y} \left[\int_{\mathbb{R}} e^{-ixy} \hat{h}_{R,2}(x)\, dx\right] f_{\xi_T}(y,\theta)\, dy \\
&= \frac{1}{2\pi} d_1 e^{-rT} \int_{\mathbb{R}} \hat{h}_{R,1}(x) \left[\int_{\mathbb{R}} e^{R_1 y - ixy} f_{\xi_T}(y,\theta)\, dy\right] dx \\
&+ \frac{1}{2\pi} d_2 e^{-rT} \int_{\mathbb{R}} \hat{h}_{R,2}(x) \left[\int_{\mathbb{R}} e^{R_2 y - ixy} f_{\xi_T}(y,\theta)\, dy\right] dx \\
&= \frac{1}{2\pi} d_1 e^{-rT} \int_{\mathbb{R}} \hat{h}_{R,1}(x) \varphi^{\theta}_{\xi_T}(-(iR_1 + x))\, dx + \frac{1}{2\pi} d_2 e^{-rT} \int_{\mathbb{R}} \hat{h}_{R,2}(x) \varphi^{\theta}_{\xi_T}(-(iR_2 + x))\, dx
\end{aligned} \tag{23}$$

On the other hand:

$$\begin{aligned}
\hat{h}_{R,1}(x) &= \int_{\mathbb{R}} e^{ixy} h_{R,1}(y)\, dy = \int_{\mathbb{R}} e^{(ix-R_1)y} h_1(y)\, dy \\
&= \int_{K_1}^{+\infty} e^{(ix-R_1)y}(y - K_1)\, dy = \int_{K_1}^{+\infty} y e^{(ix-R_1)y}\, dy - K_1 \int_{K_1}^{+\infty} e^{(ix-R_1)y}\, dy \\
&= \frac{y}{ix - R_1} e^{(ix-R_1)y}\Big|_{K_1}^{+\infty} - \frac{1}{(ix-R_1)^2} e^{(ix-R_1)y}\Big|_{K_1}^{+\infty} - \frac{K_1 e^{K_1(ix-R_1)}}{R_1 - ix} \\
&= \frac{\mathrm{e}^{K_1(ix-R_1)}\ (K_1\ (R_1 - ix) + 1)}{(R_1 - ix)^2} - K_1 \frac{\mathrm{e}^{-K_1\,(R_1 - ix)}}{R_1 - ix} \\
&= \frac{\mathrm{e}^{K_1(ix-R_1)}}{(R_1 - ix)^2}
\end{aligned}$$

Similarly:

$$\begin{aligned}
\hat{h}_{R,2}(x) &= \int_{\mathbb{R}} e^{ixy} h_{R,2}(y)\, dy = \int_{\mathbb{R}} e^{(ix-R_2)y} h_2(y)\, dy \\
&= \int_{-\infty}^{K_2} e^{(ix-R_2)y}(K_2 - y)\, dy = K_2 \int_{-\infty}^{K_2} e^{(ix-R_2)y}\, dy - \int_{-\infty}^{K_2} y e^{(ix-R_2)y}\, dy \\
&= -K_2 \frac{\mathrm{e}^{-K_2\,(R_2 - \mathrm{i}x)}}{R_2 - \mathrm{i}x} - \frac{\mathrm{e}^{-K_2(R_2 - \mathrm{i}x)}\ (K_2\ (R_2 - \mathrm{i}x) + 1)}{(R_2 - \mathrm{i}x)^2} \\
&= -\frac{\mathrm{e}^{-K_2\,(R_2 - \mathrm{i}x)}}{(R_2 - \mathrm{i}x)^2}\,(2K_2(R_2 - \mathrm{i}x) + 1)
\end{aligned}$$

The integrals in equation (23) can efficiently be calculated following a Fast Fourier Transform approach.

4. Numerical results

Daily temperature data (in Fahrenheit degrees) at the wetland Everglades(near Big Cypress Reservation), Florida, US from January 1st, 2000 to November 15th, 2017 have been collected from the NOAA National Center for Environmental Information. The data gathered yield 6245 data points. About 15% of the series values are missing, mostly around 2005. We replace them using a linear interpolation method. Observation points in the dataset consist in the average between the daily maximum and minimum temperatures.

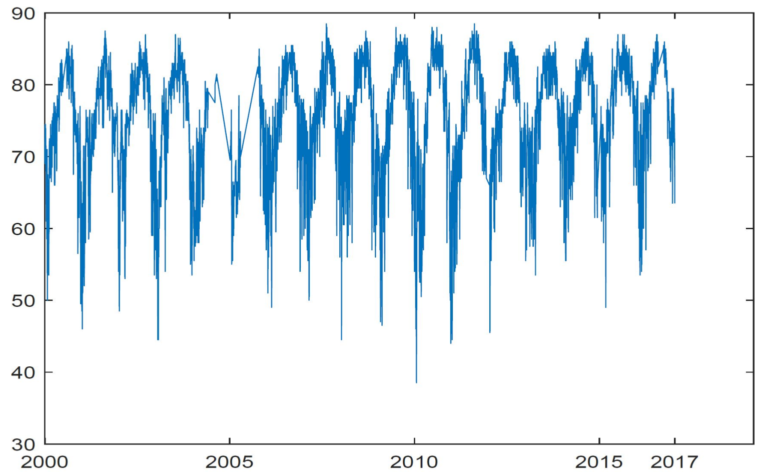


Figure 1. Historic daily average temperature in the Everglades, FL. US from 1/1/2000 to 15/11/2017

A preliminary statistical analysis of the temperature data shows the descriptive statistics as in Table 1. As can be seen, the skewness of the data is negative indicating a longer tail to the left or abrupt descents on temperatures combined with more frequent but shorter increases. The kurtosis slightly greater than three, indicating more frequent but modest movements of temperature than would be expected under assumptions of a normal distribution.

Mean	Minimum	Maximum	Std Dev	Skewness	Kurtosis
75.3271	38.5	88.5	7.9727	-0.9225	3.5578

Table 1. Statistical summary of the series of temperatures in the Everglades, 2000-2017.

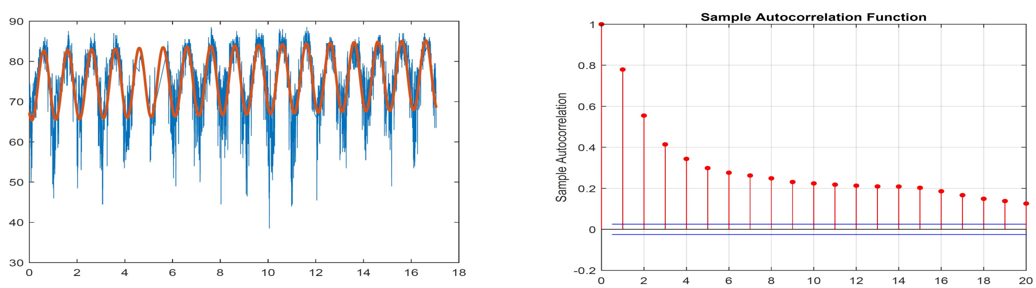
The seasonal component as described in equation (2) is adjusted via a regression model in terms of sine and cosine periodic functions with annual

	Estimate	SE	t-Stat	pValue
b_0	73.979	0.11726	630.87	0
b_1	0.15848	0.011918	13.297	8.4253e-40
b_2	-5.343	0.083029	-64.351	0
b_3	-6.818	0.082747	-82.397	0

TABLE 2. Harmonic regression fit for temperatures in the Everglades. All coefficients are statistically significant at 95%.

periodicity. In addition we consider a linear term. The results are shown in table 2.

All parameters in the regression analysis are statistically significantly. The slope in linear term indicates a slowly but important positive increase in temperature upon time. This is consistent with other climatic studies signaling the past two decades as the warmest ones since temperature is recorded. Figure 2(left) shows the harmonic fit while 2(right) reveals a significant structure of the autocorrelation function for the residuals in the regression.



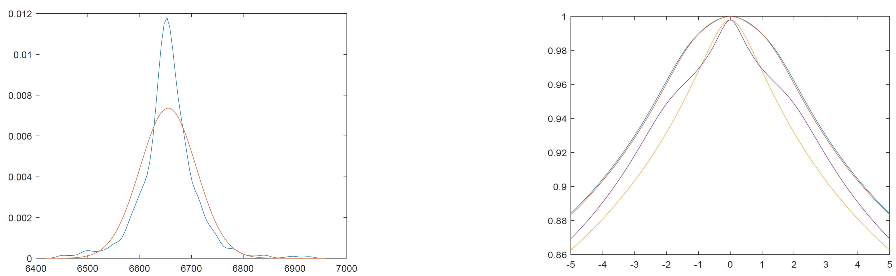
(a) (b)

FIGURE 2. Left: Seasonal trend for Everglades daily mean temperature. Right: Autocorrelation function of the residuals in the harmonic regression

In figure 3(left) the empirical p.d.f. of the CAT index during a three-month period in summer time is compared with the p.d.f. of a standard normal distribution with the same mean and variance. The p.d.f. of the CAT index shows heavier tails than the Gaussian distribution, therefore more oscillations in temperature than the expected with a normal distribution. Moreover, results of Kolmogorov-Smirnov and Anderson-Darling tests show that the temperature data do not seem to follow any of the normal, t-student, inverse Gaussian or GEV distributions distribution.

In figure 3(right) the characteristic function of the CAT index under the

Esscher EMM and different values of the parameter θ is presented. The remaining of the parameters are kept according to table 3. Notice that the true parameter θ can be computed from equation 12 following a Newton-Raphson quadrature. In fact, it depends on the parameters of the subordinator and the volatility.



(a) (b)

FIGURE 3. Left: Empirical p.d.f. of the CAT index during a three-month period in summer time is compared with the p.d.f. of a standard normal distribution with the same mean and variance. Right: Characteristic function of the CAT index under the Esscher EMM and different values of the parameter θ

An alternative pricing approach is consider a Monte Carlo simulation of the payoffs. To this end we simulate the model for temperatures introduced above with a Gamma subordinator.

param.	$a = 10$	$b = 1$	$\beta_0 = 73.979, \beta_1 = 0.16$ $\beta_2 = -1.4595, \beta_3 = -2.3897$	$\alpha = 1$	$\sigma = 5$
Contract	$T = 3/12$	$d = [20, 20]$	$K = [6300, 7200]$	$r = 0.02$	
$\theta = 0$	$ndays = 90$				

TABLE 3. Simulation parameters for temperatures WD contracts in Havana

The solution of equation (1) is obtained following an Euler-Maruyama approach. It results in the following recursive equations:

$$T_j = T_{j-1} + \Delta s_j + \alpha(s_{j-1} - T_{j-1})\Delta t + \sigma_{j-1}\Delta V_j,\ j = 1, \ldots, ndays$$

Three simulated trajectories of the temperature process during 90 summer days, starting on May 1st, are shown in figure 4. The parameters in the simulation appear in table 3. They have been chosen to match first and second empirical moments of the temperature. The maturity in the contract is three months starting May 1st, while the values $d_j = \$20$ reflects a cost of

$20 per an increase(decrease) of one degree in the temperature above(below) the strike prices. The values of d are standard in contracts traded at Chicago Mercantile Exchange. On the other hand, the strike prices have been selected accordingly to behaviors of the CAT index above and below an average historical behavior of the temperatures in the region.

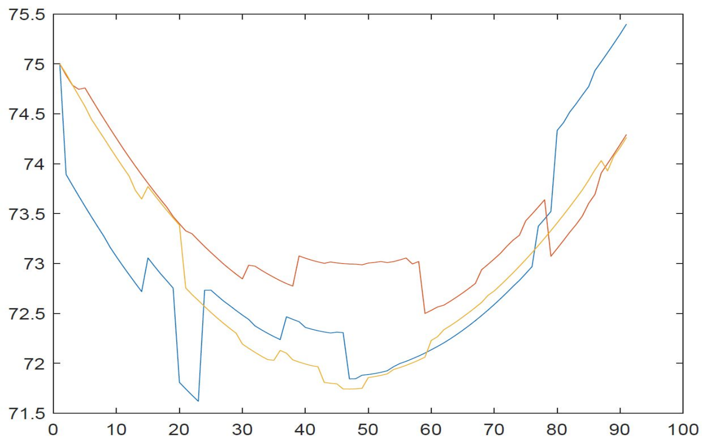


FIGURE 4. Three simulated trajectories of the temperature process during 90 days, starting on May 1st.

5. ACKNOWLEDGMENTS

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6. CONCLUSIONS

A mean-reverting time-changed Levy process with periodic mean-reverting level and volatility offers a fair model for temperatures at Everglades wetland in Florida state.
For the pricing of weather derivative contracts based on temperatures important features of the model such as the Esscher parameter under a Gerber-Shui EMM and the characteristic function of CAT indices are calculated and a methodology based on Fourier expansions is outlined. This approach provides an efficient alternative to the costly Monte Carlo simulation approach under a more realistic dynamic for the temperatures processes. In turn, the correct valuation of temperature contracts adds another tool in managing risks associated with extreme changes in climatic variables, temperatures in the present case, in critical ecological environments.

The pricing technique depends on the delicate choice of truncations in the series and the integral that have been calculated via numerical algorithms, as well as the accuracy in the estimation of the parameters. Due to the existence of multiple parameters in the model the estimation is a challenging problem beyond the scope of the paper. Nonetheless, it can be pointed out that the use of an Esscher transform in computing the EMM offers an important practical advantage, as historic data can be used to the task. A calibration based on the market price of the contracts results problematic because of its liquidity and local dependence.

7. Appendix

A Fourier expansion of the p.d.f. $f_{\xi_T}(x,\theta)$ on an interval $[b_1,b_2]$ is given by:

$$f_{\xi_n}(x,\theta) \quad = \quad \sum_{k=0}^{+\infty} A_k(\theta) cos\left(k\pi\frac{x-b_1}{b_2-b_1}\right) \tag{24}$$

where the coefficients in the expansion, the first of them divided by two, are:

$$\begin{aligned} A_k(\theta) &= \frac{2}{b_2-b_1}\int_{b_1}^{b_2} f_{\xi_T}(y,\theta) cos\Big(k\pi\frac{y-b_1}{b_2-b_1}\Big)\,dy \\ &= \frac{2}{b_2-b_1}\int_{b_1}^{b_2} f_{\xi_T}(y,\theta) Re\Big(e^{ik\pi\frac{y-b_1}{b_2-b_1}}\Big)\,dy \\ &= \frac{2}{b_2-b_1} Re\Big(\int_{b_1}^{b_2} f_{\xi_T}(y,\theta) e^{ik\pi\frac{y-b_1}{b_2-b_1}}\,dy\Big) \\ &\simeq \frac{2}{b_2-b_1} exp\left(-i\frac{k\pi b_1}{b_2-b_1}\right)\varphi^{\theta}_{\xi_T}\Big(\frac{k\pi}{b_2-b_1}\Big) \end{aligned} \tag{25}$$

for b_1 and b_2 large enough.
Replacing (25) into (24), then (24) in (18) we have after truncating the series at N_1 terms and the change of variable $y = x - K_1$:

$$\begin{aligned} I_1 &\simeq \sum_{k=0}^{+\infty} A_k(\theta)\int_{b_1}^{b_2}(x-K_1)_+ cos\left(k\pi\frac{x-b_1}{b_2-b_1}\right)\,dx \\ &= \sum_{k=0}^{+\infty} A_k(\theta)\int_{b_1-K_1}^{b_2-K_1} y_+ cos\left(k\pi\frac{y-b_1+K_1}{b_2-b_1}\right)\,dy \\ &\simeq \sum_{k=0}^{N_1} A_k(\theta)\int_{b_4}^{b_5} y\ cos\left(k\pi\frac{y-b_3}{b_5-b_3}\right)\,dy \end{aligned}$$

Furthermore, for $k > 0$:

$$\begin{aligned}\int_{b_4}^{b_5} y\ cos\left(k\pi\frac{y-b_3}{b_5-b_3}\right)\ dy &= \frac{(b_3-b_5)b_4}{k\pi}\sin\left(k\pi\frac{b_4-b_3}{b_5-b_3}\right)\\ &+ \left(\frac{b_5-b_3}{k\pi}\right)^2\left((-1)^k-\cos\left(k\pi\frac{b_4-b_3}{b_5-b_3}\right)\right)\end{aligned}$$

Then, separating the first term in the summation:

$$\begin{aligned}I_1 &= \frac{b_5^2-b_4^2}{2(b_5-b_3)}+2\sum_{k=1}^{N_1} exp\left(-i\frac{k\pi b_1}{b_5-b_3}\right)\varphi^{\theta}_{\xi_T}\left(\frac{k\pi}{b_5-b_3}\right)\\ &\left(-\frac{b_4}{k\pi}\sin\left(k\pi\frac{b_4-b_3}{b_5-b_3}\right)+\frac{b_5-b_3}{k^2\pi^2}\left((-1)^k-\cos\left(k\pi\frac{b_4-b_3}{b_5-b_3}\right)\right)\right)\end{aligned}$$

In a similar analysis:

$$\begin{aligned}\int_{b_6}^{b_7} y\ cos\left(k\pi\frac{y-b_8}{b_8-b_7}\right)\ dy &= \frac{(b_7-b_8)b_6}{k\pi}\sin\left(k\pi\frac{b_6-b_8}{b_8-b_7}\right)\\ &+ \left(\frac{b_8-b_7}{k\pi}\right)^2\left((-1)^k-\cos\left(k\pi\frac{b_6-b_8}{b_8-b_7}\right)\right)\end{aligned}$$

Hence:

$$\begin{aligned}I_2 &\simeq \sum_{k=0}^{N_1} A_k(\theta)\int_{b_1}^{b_2}(K_2-x)_+ cos\left(k\pi\frac{x-b_1}{b_2-b_1}\right)\ dx\\ &= \sum_{k=0}^{N_1} A_k(\theta)\int_{K_2-b_2}^{K_2-b_1} y_+ cos\left(k\pi\frac{K_2-y-b_1}{b_2-b_1}\right)\ dy\\ &= \frac{b_7^2-b_6^2}{2(b_8-b_7)}+2\sum_{k=1}^{N_1} exp\left(-i\frac{k\pi b_1}{b_8-b_7}\right)\varphi^{\theta}_{\xi_T}\left(\frac{k\pi}{b_8-b_7}\right)\\ &\left(-\frac{b_6}{k\pi}\sin\left(k\pi\frac{b_6-b_8}{b_8-b_7}\right)\right.\\ &+ \left.\frac{b_8-b_7}{k^2\pi^2}\left((-1)^k-\cos\left(k\pi\frac{b_6-b_8}{b_8-b_7}\right)\right)\right)\end{aligned}$$

References

[1] Benth, F. and Sgarra, C. (2009)The Risk Premium and the Esscher Transform in Power Markets. Stochastic Analysis and Applications, vol.30(1).

[2] F. E. Benth and J. Saltyte-Benth. Stochastic modelling of temperature variations with a view towards weather derivatives. Applied Mathematical Finance, 12(1):53-85, 2005.

[3] Carr P. and Madan D.(1999)Option valuation using the fast Fourier transform. Journal of Computational Finance, vol.2, no.4, pg.61-73

[4] Fang and Cornelis W. Oosterlee (2008). A novel pricing method for European options based on Fourier-cosine series expansion. *SIAM Journal of Scientific Computing.*

[5] Gerber, H. U. and Shiu, E. S. W. (1994) Option pricing by Esscher-transforms. Transactions of the Society of Actuaries 46, 99-191.

[6] Kallsen Jan and Muhle-Karbe Johannes(2011). Method of moment estimation in time-changed Levy models, Statistics and Risk Modeling, De Gruyter, vol. 28(2), pages 169-194, May.

[7] Porthiyas, J. (2019) Pricing Weather Contracts Under Mean-reverting Levy Processes. M Sc. Thesis. Department of Mathematics. Ryerson University.

[8] Raible, S.(2000) Levy processes in finance: Theory, numerics, and empirical facts. Tech. rep., PhD thesis, Universit¨at Freiburg.

[9] Chunfa Wang (2017). Pricing European Options by Stable Fourier-Cosine Series Expansions https://arxiv.org/abs/1701.00886

[10] Samuel Asante Gyamerah, Philip Ngare, and Dennis Ikpe (2018)Regime-Switching Temperature Dynamics Model for Weather Derivatives. International Journal of Stochastic Analysis. Volume 2018, Article ID 8534131, 15 pages. https://doi.org/10.1155/2018/8534131

[11] A. Swishchuk and K. Cui (2013) Weather derivatives with applications to canadian data. Journal of Mathematical Finance, 3(1):81-95, 2013.

[12] Bowen Zhang and Cornelis W. Oosterlee. Efficient pricing of european-style asian optoins under exponential *SIAM Journal of Financial Mathematics*, 2013. Lévy processes based on Fourier cosine expansions.

[13] Didier Dacunha-Castelle, Thi Thu Huong Hoang and Sylvie Parey (2015). Modeling of air temperatures: preprocessing and trends, reduced stationary process, extremes, simulation Journal de la Socièté Française de Statistique Vol. 156 No. 1.