CHARACTERISTIC FUNCTION AND ESSCHER TRANSFORM OF A SWITCHING LEVY MODEL FOR THE TEMPERATURE DYNAMIC

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Abstract. In this paper we extend the models in [1, 9, 10] for the dynamic of the temperatures by considering random switchings between Levy noises instead of Brownian motions, with a mean-reverting movement towards a seasonal periodic function. The use of Levy noises allows for jumps, capturing, together with the regime changes, sudden and relatively persistent oscillations in the weather. An approximated close-form expression for the characteristic function of the temperature process under an Esscher transform is given.

1. Introduction

In [1, 9, 10] switching models for temperatures have been proposed. In such models the temperature evolves between mean-reverting stochastic differential equations whose background noises switch at random times between Brownian motions with different volatilities. We extend these models by considering random switchings between Levy noises with a mean-reverting movement towards a seasonal periodic function. The use of Levy noises and regime-switching models allow random jumps in the underlying temperature process, capturing sudden and relatively persistent oscillations in temperatures and the weather in general.

With a view in the pricing weather derivatives we find the characteristic function of integrals with respect to Levy switching processes by conditioning on the number of switches over a given time interval and investigate the choice of an equivalent martingale measure(EMM) to create a risk-neutral setting with the use of the Esscher transformation, see [5]. We implement the findings in two time-changed Levy models driven by inverse Gaussian and Gamma subordinators.

In the context of discrete-time regime switching models have been successfully considered since the pioneer work of [4]. Continuous-time switching Levy models have been recently introduced in [3] in connection with the modeling of oil prices.

The organization of the paper is the following:

In section 2 we present the switching Levy model for temperature and compute the characteristic function of the temperatures under the historic measure. In section 3 we find the characteristic function of the temperatures under the measure generate by an Esscher transform, as long as the selection of the parameter in the Esscher transform to obtain an EMM risk-neutral measure. In section 4 we discuss the cases of time-changed Levy models with inverse Gaussian and Gamma subordinators while in section 5 we conclude.

2. A switching Levy model for temperature

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space verifying the usual conditions. For a stochastic process $(X_t)_{t\geq 0}$ defined on the space filtered space above the functions φ_{X_t} and $l_{X_t}(u) = \log \varphi_{X_t}(-iu)$ represent its characteristic function and the log-cumulant generating function respectively. When the process has stationary and independent increments the later does not depend on t.
The class $\sigma_{s,t}(\tau)$ as the σ-algebra generated by the random variables $\{\tau_s, \tau_{s+1}, \ldots, \tau_t\}$. In particular, we write $\sigma_t(\tau) := \sigma_{0,t}(\tau)$.
The discounted process $(\tilde{X}_t)_{t\geq 0}$ is defined as $\tilde{X}_t = e^{-rt}X_t$, where $r > 0$ is a constant interest rate.
Furthermore, $M(a, b, z)$ denotes the Kummer's confluent hypergeometric function:

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}\, du$$

Also, $\gamma(a, z)$ is the incomplete Gamma function:

$$\gamma(a,z) = \int_0^z x^{a-1}e^{-x}\, dx,\ a \geq 0$$

Let $(T_t)_{t\geq 0}$ be a stochastic process where T_t is the temperature of certain region at time t. The dynamic of the temperature follows a switching between two regimes. The regimes are driven by a stationary Markov process $\{r_t\}_{t\geq 0}$ with values in the space $E_r = \{1, 2\}$.
Moreover, under regime j the temperature process $(T_t^j)_{t\geq 0}$ verifies the mean-reverting stochastic differential equation:

$$dT_t^j = ds_t + \alpha^j(s_t - T_t^j)dt + \sigma^j dV_t^j \quad (1)$$

where the temperature reverts to the deterministic seasonal process $(s_t)_{t\geq 0}$ given by:

$$s_t = \beta_0 + \beta_1 t + \beta_2 \sin\left(\frac{2\pi}{365}t\right) + \beta_3 \cos\left(\frac{2\pi}{365}t\right) \quad (2)$$

The parameters $\alpha^j > 0$ are the mean-reversion rates under regimes $j = 1, 2$ while σ^j are their respective volatilities or standard deviations. The parameters in the seasonal function may also depend on the regime. For simplicity we consider the case of constant coefficients.
On the other hand, the dynamic of the regime-switching Markov process is given by the following transition probabilities:

$$\begin{aligned} P\{r_{t+h} = j/r_t = k\} &= \lambda_{jk}h + o(h),\ j \neq k \\ P\{r_{t+h} = j/r_t = j\} &= -\lambda_{jj}h + o(h) \end{aligned}$$

Without loss of generality we assume the initial conditions $r_0 = 1$, i.e. the process starts under regime 1. The analysis of a process starting under regime 2 or at a random initial value is similar.
Let N_t be the number of transitions between regimes in the interval $[0, t)$.
Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of stopping times representing the instants where the system changes of regime. Notice that $\tau_0 = 0$ and $\tau_{N_t} \leq t$. Set also $v_j = \tau_j - \tau_{j-1}$ as the times between two consecutive changes of regime.

Remark 1. Because of the Markovian nature of the switching process the random variables v_j are independent such that $v_j \sim exp(\lambda_{12})$ if the transition occurs from regime 1 to regime 2 and $v_j \sim exp(\lambda_{21})$ otherwise.

The solution of equation (1) is given in the next elementary result. It follows from Ito lemma.

Lemma 2. The solution of equation (1) is:

$$T_t = C_1(t,\alpha) + \sigma e^{-\alpha t} W_t \tag{3}$$

with

$$W_t = \sum_{j=1}^{N_t} \int_{\tau_{j-1}}^{\tau_j} e^{\alpha u} dV_u^{\nu(j)} + \int_{\tau_{N_t}}^{t} e^{\alpha u} dV_u^{\nu(N_t+1)} \tag{4}$$

where $\nu(j) = \frac{3+(-1)^j}{2}$ and $C_1(t,\alpha) = s_t + e^{-\alpha t}(T_0 - s_0)$

A result about the probability distribution of the instants and the number of regime changes in $[0,t)$ is given in the next two lemmas. To this end we introduce the following quantities:

$$\begin{aligned} D(m,n,l) &= \frac{\lambda_{12}^m}{\lambda_{21}^n \Gamma(l)},\ l \geq 1 \\ D(m,n,0) &= 0,\ C(m,n,0) = 1 \\ C(m,n,l) &= \frac{m^{(l)}(1-\frac{\lambda_{12}}{\lambda_{21}})^l}{n^{(l)} l!} = \frac{\binom{m+l-1}{l}(1-\frac{\lambda_{12}}{\lambda_{21}})^l}{\binom{n+l-1}{l} l!},\ l > 0 \end{aligned}$$

where $a^{(l)}$ is defined as the Pochhammer symbol or raising factorial, $a^{(0)} = 1, a^{(l)} = a(a+1)\ldots(a+l-1)$.
Hence:

Lemma 3. Let τ_l be the time of the l-th regime change, f_{τ_l} and F_{τ_l} the respective probability density function (p.d.f.) and cumulative distribution function(c.d.f.). Then:

$$\begin{aligned} F_{\tau_{2k}}(t) &= D(k,k,2k) \sum_{l=0}^{+\infty} C(k,2k,l) \gamma(2k+l, \lambda_{21} t),\ k \geq 1 \\ F_{\tau_{2k+1}}(t) &= D(k+1,k+1,2k+1) \sum_{l=0}^{+\infty} C(k+1,2k+1,l) \gamma(2k+l+1, \lambda_{21} t) \end{aligned}$$

Proof. First, define the independent random variables:

$$X_1 = \sum_{l=1}^{k} v_{2l-1},\ X_2 = \sum_{l=1}^{k} v_{2l}$$

Notice that $\tau_{2k} = X_1 + X_2$, where $X_1 \sim Erlang(k, \lambda_{12})$ and $X_2 \sim Erlang(k, \lambda_{21})$. The probability distribution of the sum of two independent Erlang random variables with different shape and rate parameters, or equivalently the sum of independent exponential random variables with different rates, has been found in [7] or in [6] for example. Adapting their results, specifically corollary 6.3 in [6], to our case we have:

$$f_{\tau_{2k}}(x) = D(k,-k,2k) M(k,2k,(\lambda_{21}-\lambda_{12})x) x^{2k-1} e^{-\lambda_{21} x},\ x \geq 0,\ \lambda_{21} > \lambda_{12}$$

On the other hand, the confluent hypergeometric function can be expanded as:

$$M(a,b,z) = \sum_{l=0}^{+\infty} \frac{a^{(l)}}{b^{(l)} l!} z^l$$

See [8] for definition and property of the function above.
Therefore for $t \geq 0$:

$$\begin{aligned} F_{\tau_{2k}}(t) &= D(k,k,2k) \int_0^t x^{2k-1} e^{-\lambda_{21} x} M(k,2k,(\lambda_{21}-\lambda_{12})x)\, dx \\ &= D(k,k,2k) \sum_{l=0}^{+\infty} C(k,2k,l) \int_0^t x^{2k+l-1} e^{-\lambda_{21} x}\, dx \\ &= D(k,k,2k) \sum_{l=0}^{+\infty} C(k,2k,l) \gamma(2k+l, \lambda_{21} t) \end{aligned}$$

Similarly $\tau_{2k+1} = Y_1 + Y_2$, where $Y_1 \sim Erlang(k+1, \lambda_{12})$ and $Y_2 \sim Erlang(k, \lambda_{21})$ and independent random variables. Then:

$$f_{\tau_{2k+1}}(x) = D(k+1,-k,2k+1) M(k+1,2k+1,(\lambda_{21}-\lambda_{12})x) x^{2k} e^{-\lambda_{21} x}$$

which leads to:

$$\begin{aligned} F_{\tau_{2k+1}}(t) &= D(k+1,k+1,2k+1) \int_0^t x^{2k} e^{-\lambda_{21} x} M(k+1,2k+1,(\lambda_{21}-\lambda_{12})x)\, dx \\ &= D(k+1,k,2k+1) \sum_{l=0}^{+\infty} C(k+1,2k+1,l) \gamma(2k+l+1, \lambda_{21} t) \end{aligned}$$

□

Lemma 4. Let N_t be the number of transitions between regimes in the interval $[0,t)$. Then, for $k \in \mathbb{N}$:

$$P(N_t = k) = F_{\tau_k}(t) - F_{\tau_{k+1}}(t) \tag{5}$$

Proof. We have:

$$P(N_t = 2k) = P(\tau_{2k} < t, \tau_{2k+1} > t) = P(\tau_{2k} < t) - P(\tau_{2k+1} < t) = F_{\tau_{2k}}(t) - F_{\tau_{2k+1}}(t)$$

Similarly:

$$P(N_t = 2k+1) = F_{\tau_{2k+1}}(t) - F_{\tau_{2k+2}}(t)$$

□

We compute the characteristic function of some integrals of the background noise process given by the switching Levy process $(V_t)_{t\geq 0}$. To this end we will extend a well-known result about functional of a Levy process $(\xi_t)_{t\geq 0}$ with $\xi_0 = 0$ and a measurable function f, see for example [2]. Namely:

$$E[exp(i \int_0^t f(s)\, dX_s)] = exp(\int_0^t l_X(-if(s))\, ds) \tag{6}$$

where $(X_t)_{t\geq 0}$ is a Levy process with $X_0 = 0$ and f is a measurable function.
The main result in the section is given below.

Theorem 5. Let $(\xi_t)_{t\geq 0}$ be a two-regime switching Levy process starting at regime one, with log-cumulant generating function l_{ξ^j} when the process is at regime $l=1,2$. Then,

$$E[exp(i\int_0^t f(s)\,d\xi_s)] = \left(\sum_{k=0}^{+\infty} M_1(k)[P(N_t=2k)+\lambda_{12}J_1^k P(N_t=2k+1)]\right) \left(\sum_{k=0}^{+\infty} J_3(t,k)P(N_t=2k)+\sum_{k=0}^{+\infty} J_4(t,k)P(N_t=2k+1)\right) \tag{7}$$

where:

$$\begin{aligned}
J_l &= \int_0^{+\infty} e^{I_l(x)}e^{-\lambda_l x}\,dx,\ \ l=1,2\\
J_3(t,k) &= D(k,-k,2k)\sum_{l=0}^{+\infty} C(k,2k,l)G_1(t,2k+l)\lambda_2^l\\
J_4(t,k) &= D(k+1,-k,2k+1)\sum_{l=0}^{+\infty} C(k+1,2k+1,l)G_2(t,2k+l+1)\lambda_2^l\\
G_j(t,m) &= \int_0^t e^{I_j(t-z)}z^{m-1}e^{-\lambda_{21}z}\,dz,\ j=1,2\\
M_1(k) &= \lambda_1^k\lambda_2^k J_1^k J_2^k
\end{aligned}$$

and

$$I_j(x)=\int_0^x l_{\xi^j}(-if(s))\,ds,\ \ j=1,2$$

Expressions for $P(N_t=2k)$ and $P(N_t=2k+1)$ are given in the previous lemma.

Proof. For a measurable function f we write:

$$W_t(f) = \sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(u)dV_u^{\nu(j)}+\int_{\tau_{N_t}}^t f(u)dV_u^{N_t+1} \tag{8}$$

Now, let $\sigma_t(\tau)$ be the σ-algebra generated by the random variables $(\tau_n)_{n\in\mathbb{N}}, n\leq\tau_{N_t}$. Notice that between times of regime changes the process $(\xi_t)_{t\geq 0}$ is a Levy process. Hence, equation (6) applies.
Conditioning on $N_t=2k$ and $\sigma_t(\tau)$ we have, from the independence and the stationarity of the increments of a Levy process, that:

$$\begin{aligned}
&E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\,d\xi_s^{\nu(j)}/N_t=2k,\sigma_t(\tau))] = E[exp(i\sum_{j=1}^{2k}\int_{\tau_{j-1}}^{\tau_j} f(s)\,d\xi_s^{\nu(j)}/\sigma_t(\tau))]\\
=&\prod_{j=1}^{2k}E[exp(i\int_{\tau_{j-1}}^{\tau_j} f(s)\,d\xi_s^{\nu(j)}/\sigma_t(\tau))] = \prod_{j=1}^{2k}E[exp(i\int_0^{v_j} f(s)\,d\xi_s^{\nu(j)}/\sigma_t(\tau))]\\
=&exp[\sum_{j=1}^{2k}\int_0^{v_j} l_{\xi^{\nu(j)}}(-if(s))\,ds/\sigma_t(\tau)] = exp[\sum_{j=1}^{2k} I_{\nu(j)}(v_j)]
\end{aligned}$$

and

$$\begin{aligned} & E[exp(i\int_{\tau_{N_t}}^t f(s)d\xi_s^1/N_t = 2k, \sigma_t(\tau))] = E[exp(i\int_{\tau_{2k}}^t f(s)\ d\xi_s^1/\sigma_t(\tau))] \\ = & E[exp(i\int_0^{t-\tau_{2k}} f(s)\ d\xi_s^1)/\sigma_t(\tau)] \\ = & exp[\int_0^{t-\tau_{2k}} l_{\xi^1}(-if(s))\ ds] = exp[I_1(t-\tau_{2k})] \end{aligned}$$

By similar analysis, conditioning on $[N_t = 2k+1]$, we have:

$$\begin{aligned} & E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)})/N_t = 2k+1, \sigma_t(\tau)] = E[exp(i\sum_{j=1}^{2k+1}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)}/\sigma_t(\tau))] \\ = & \prod_{j=1}^{2k+1} E[exp(i\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)}/\sigma_t(\tau))] \\ = & \prod_{j=1}^{2k+1} E[exp(i\int_0^{v_j} f(s)\ d\xi_s^{\nu(j)}/\sigma_t(\tau))] \\ = & exp[\sum_{j=1}^{2k+1}\int_0^{v_j} l_{\xi^{\nu(j)}}(-if(s))\ ds/\sigma_t(\tau)] \\ = & exp[\sum_{j=1}^{2k+1} I_{\nu(j)}(v_j)] \end{aligned}$$

and

$$\begin{aligned} & E[exp(i\int_{\tau_{N_t}}^t f(s)d\xi_s^1)/N_t = 2k+1, \sigma_t(\tau)] = E[exp(i\int_{\tau_{2k+1}}^t f(s)\ d\xi_s^2)/\sigma_t(\tau)] \\ = & E[exp(i\int_0^{t-\tau_{2k+1}} f(s)\ d\xi_s^2)/\sigma_t(\tau)] \\ = & exp[\int_0^{t-\tau_{2k+1}} l_{\xi^1}(-if(s))\ ds] = exp[I_2(t-\tau_{2k})] \end{aligned}$$

If there are $2k$ changes of regime on the interval $[0,t)$ there will be k subintervals where the process is at regime 1 and another k where the process is at regime 2. The remaining time on $[0,t)$, i.e. during the interval $[t_{N_t}, t)$, the process is at regime 1.

In a similar analysis, if there are $2k+1$ changes of regime on the interval $[0,t)$ there will be $k+1$ subintervals where the process is at regime 1 and another k where the process is at regime 2. The process remains in regime 2 during $[\tau_{N_t}, t)$.

On the other hand, we have:

$$F_{t-\tau_{N_t}}(x) = P(\tau_{N_t} \geq t-x) = 1 - F_{\tau_{N_t}}(t-x),\ x < t$$

Then, conditionally on the event $[N_t = 2k]$ we have respectively:

$$f_{t-\tau_{2k}}(x) \quad = \quad f_{\tau_{2k}}(t-x) = D(k,-k,2k)(t-x)^{2k-1}e^{-\lambda_{21}(t-x)}M(k,2k,(\lambda_{21}-\lambda_{12})(t-x))$$

Then, conditionally on $[N_t = 2k+1]$:

$$f_{t-\tau_{2k+1}}(x) \quad = \quad f_{\tau_{2k+1}}(t-x) = D(k+1,-k,2k+1)(t-x)^{2k}e^{-\lambda_{21}(t-x)}M(k+1,2k+1,(\lambda_{21}-\lambda_{12})(t-x))$$

for $0 \leq x \leq t$ and zero otherwise.
Furthermore:

$$\begin{aligned}
& E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)}))/\sigma_t(\tau)] \\
= & \sum_{k=0}^{+\infty} E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)}))/N_t = 2k, \sigma_t(\tau)]P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty} E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)}))/N_t = 2k+1, \sigma_t(\tau)]P(N_t = 2k+1) \\
= & \sum_{k=0}^{+\infty} exp[\sum_{j=1}^{2k} I_{\nu(j)}(v_j)]P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty} exp[\sum_{j=1}^{2k+1} I_{\nu(j)}(v_j)]P(N_t = 2k+1)
\end{aligned}$$

and

$$\begin{aligned}
& E[exp(i\int_{\tau_{N_t}}^{t} f(s) d\xi_s^1)/\sigma_t(\tau)] \\
= & \sum_{k=0}^{+\infty} E[exp(i\int_{\tau_{N_t}}^{t} f(s) d\xi_s^1)/N_t = 2k, \sigma_t(\tau)]P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty} E[exp(i\int_{\tau_{N_t}}^{t} f(s) d\xi_s^2)/N_t = 2k+1, \sigma_t(\tau)]P(N_t = 2k+1) \\
= & \sum_{k=0}^{+\infty} exp[I_1(t-\tau_{2k})]P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty} exp[I_2(t-\tau_{2k+1})]P(N_t = 2k+1)
\end{aligned}$$

Hence:

$$\begin{aligned}
& E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)})] = E[E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)})/\sigma_t(\tau)]] \\
= & \sum_{k=0}^{+\infty}\prod_{j=1}^{2k} E[exp(I_{\nu(j)}(v_j))]P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty}\prod_{j=1}^{2k+1} E[exp(I_{\nu(j)}(v_j))]P(N_t = 2k+1) \\
= & \sum_{k=0}^{+\infty}(E[exp(I_1(v))])^k(E[exp(I_2(v^*))])^k P(N_t = 2k) \\
+ & \sum_{k=0}^{+\infty}(E[exp(I_1(v))])^{k+1}(E[exp(I_2(v^*))])^k P(N_t = 2k+1)
\end{aligned}$$

where $v \sim exp(\lambda_{12})$ is the time between changes from regime 1 to regime 2 and $v* \sim exp(\lambda_{21})$ is the time between changes from regime 2 to regime 1. Also,

$$\begin{aligned} & E[exp(i\int_{\tau_{N_t}}^{t} f(s)d\xi_s^1)] = E[E[exp(i\int_{\tau_{N_t}}^{t} f(s)d\xi_s^1)/\sigma_t(\tau)]] \\ = & \sum_{k=0}^{+\infty} E[exp(I_1(t-\tau_{2k}))]P(N_t = 2k) \\ + & \sum_{k=0}^{+\infty} E[exp(I_2(t-\tau_{2k+1}))]P(N_t = 2k+1) \end{aligned}$$

Moreover:

$$\begin{aligned} E[e^{I_1(v)}] &= \lambda_{12}\int_0^{+\infty} e^{I_1(x)-\lambda_{12}x}\,dx := \lambda_{12}J_1 \\ E[e^{I_2(v*)}] &= \lambda_{21}\int_0^{+\infty} e^{I_2(x)-\lambda_{21}x}\,dx := \lambda_{21}J_2 \end{aligned}$$

On the other hand:

$$\begin{aligned} E[e^{I_1(t-\tau_{2k})}] &= \int_0^t e^{I_1(x)} f_{t-\tau_{2k}}(x)\,dx \\ &= D(k,-k,2k)\int_0^t e^{I_1(x)}(t-x)^{2k-1}e^{-\lambda_{21}(t-x)}M(k,2k,(\lambda_{21}-\lambda_{12})(t-x))\,dx \end{aligned}$$

The previous equation, after the change of variable $z = t - x, dz = -dx$ reads:

$$\begin{aligned} &= D(k,-k,2k)\int_0^t e^{I_1(t-z)}z^{2k-1}e^{-\lambda_{21}z}M(k,2k,(\lambda_{21}-\lambda_{12})z)\,dz \\ &= D(k,-k,2k)\sum_{l=0}^{+\infty}\frac{k^{(l)}(\lambda_{21}-\lambda_{12})^l}{(2k)^{(l)}l!}\int_0^t e^{I_1(t-z)}z^{2k+l-1}e^{-\lambda_{21}z}\,dz \\ &= D(k,-k,2k)\sum_{l=0}^{+\infty} C(k,2k,l)G_1(t,2k+l)\lambda_2^l := J_3(t,k) \end{aligned}$$

Similarly:

$$\begin{aligned} E[e^{I_2(t-\tau_{2k+1})}] &= \int_0^t e^{I_2(x)} f_{t-\tau_{2k+1}}(x)\,dx \\ &= D(k+1,-k,2k+1)\int_0^t e^{I_2(x)}(t-x)^{2k}e^{-\lambda_{21}(t-x)}M(k+1,2k+1,(\lambda_{21}-\lambda_{12})(t-x))\,dx \\ &= D(k+1,-k,2k+1)\sum_{l=0}^{+\infty}\frac{(k+1)^{(l)}(\lambda_{21}-\lambda_{12})^l}{(2k+1)^{(l)}l!}\int_0^t e^{I_2(t-z)}z^{2k+l}e^{-\lambda_{21}z}\,dz \\ &= D(k+1,-k,2k+1)\sum_{l=0}^{+\infty} C(k+1,2k+1,l)G_2(t,2k+l+1)\lambda_2^l := J_4(t,k) \end{aligned}$$

Finally by the independence of the process on non-overlapping intervals:

$$E[exp(i\int_0^t f(s)\ d\xi_s)] = E[exp(i\sum_{j=1}^{N_t}\int_{\tau_{j-1}}^{\tau_j} f(s)\ d\xi_s^{\nu(j)})]E[exp(i\int_{\tau_{N_t}}^{t} f(s)d\xi_s^1)]$$

$$= \left(\sum_{k=0}^{+\infty}(E[exp(I_1(v))])^k(E[exp(I_2(v^*))])^k P(N_t = 2k)\right.$$

$$\left. + \sum_{k=0}^{+\infty}(E[exp(I_1(v))])^{k+1}(E[exp(I_2(v^*))])^k P(N_t = 2k+1)\right)$$

$$\left(\sum_{k=0}^{+\infty} E[exp(I_1(t-\tau_{2k}))]P(N_t = 2k) + \sum_{k=0}^{+\infty} E[exp(I_2(t-\tau_{2k+1}))]P(N_t = 2k+1)\right)$$

from which equation (7) immediately follows. □

The following result describes the characteristic function of the temperature process under the historic measure P.

Proposition 6. For the model described by equations (1), (2) and (12) the characteristic function of T_t under the probability P is:

$$\varphi_{T_t}(u) = exp(iuC_1(t,\alpha))\varphi_{W_t}(u\sigma e^{-\alpha t}) \tag{9}$$

where the characteristic function of W_t, denoted by $\varphi_{W_t}(u)$, is given by equation (7) in Theorem 5 applied to $\xi_t^l = V_t^l, f(s) = u\sigma e^{-\alpha(t-s)}$ and

$$I_j(x) := I_j(x,u) = \int_0^x l_{R^j}(-iu\mu_1^j\sigma e^{-\alpha(t-s)} - \frac{1}{2}u^2\sigma^2 e^{-2\alpha(t-s)})\ ds,\ \ j = 1,2$$

Proof. First, notice that:

$$\begin{aligned}
\varphi_{V_t^j}(u) &= E[E[\exp(i(uV_t^j)/R_t^j)]] = E[\exp(iu\mu_1^j R_t^j)E[\exp(iuB_{R_t^j}/R_t^j)]] \\
&= E[\exp(iu\mu_1^j R_t^j)\exp(-\frac{1}{2}R_t^j u^2)] = E[\exp(i(u\mu_1^j + \frac{1}{2}iu^2)R_t^j)] \\
&= \varphi_{R_t^j}(u\mu_1^j + \frac{1}{2}iu^2)
\end{aligned}$$

Hence:

$$l_{V^j}(u) = l_{R^j}(u\mu_1^j + \frac{1}{2}u^2),\ j = 1,2. \tag{10}$$

By Lemma 1 and equation (6):

$$\begin{aligned}
\varphi_{T_t}(u) &= E[e^{iuT_t}] = exp(iuC_1(t,\alpha))E[exp(iu\sigma e^{-\alpha t}W_t)] \\
&= exp(iuC_1(t,\alpha))E[exp(iu\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dV_s)] = exp(iuC_1(t,\alpha))\varphi_{W_t}(u\sigma e^{-\alpha t})
\end{aligned}$$

□

3. Switching model under an Esscher transformation

In order to select the EMM for pricing purposes we consider an Esscher transform of the historic measure P. See [5] for a rationale in terms of a utility-maximization wealth criterion.
Thus, for a stochastic process $(X_t)_{t\geq 0}$ we consider the change of probability:

$$\frac{d\mathcal{Q}_t^\theta}{dP_t} = \exp(\theta X_t - l_{X_t}(\theta)),\ \theta \in \mathbb{R} \tag{11}$$

where P_t and $\mathcal{Q}_t^\theta$ are the respective restrictions of P and $\mathcal{Q}^\theta$ to the σ-algebra $\mathcal{F}_t$. By $\varphi_{X_t}^\theta(u)$ and $l_X^\theta(u)$ are defined respectively as the characteristic function and cumulant generating function of a process $(X_t)_{t\geq 0}$ under the probability $\mathcal{Q}^\theta$ obtained by an Esscher transformation as given in equation (11).
For consistency we denote $\varphi_{X_t}^0 := \varphi_{X_t}$ and $l_X^0 = l_X$.
By analogy with the case of financial underlying assets the risk-neutral measure $\mathcal{Q}^\theta$ making the discounted temperatures process $(\tilde{T}_t)_{t\geq 0}$ a martingale. The expected value under $\mathcal{Q}^\theta$ is denoted E_θ.
Consider switching subordinator processes $(R_t^j)_{t\geq 0}$ corresponding to regime j, for $j = 1,2$ and time-changed processes $(V_t^l)_{t\geq 0}$ introduced as:

$$V_t^j \quad = \quad B_{R_t^j} + \mu_1^j R_t^j,\ j = 1,2 \tag{12}$$

Here $\mu_1^j \in \mathbb{R}, j = 1,2.$ is a parameter in the model and $(B_t)_{t\geq 0}$ is a standard Brownian motion. It is assumed both subordinators to have finite moments of some convenient order depending of the specific process chosen.
Under an Esscher transform of parameter θ the probabilities of the number of changes are:

$$P^\theta(N_t = k) \quad = \quad \frac{e^{\theta k} P(N_t = k)}{M_{N_t}(\theta)} \tag{13}$$

where $M_{N_t}(\theta) = E[e^{\theta N_t}]$ its moment generating function(m.g.f.) of N_t.
The next result provides an expression for the characteristic function under an Esscher transformation of parameter θ.

Theorem 7. Under the probability $\mathcal{Q}^\theta$ defined by the Esscher transform in equation (11) and the condition $\lambda_{12}, \lambda_{21} < \theta$ the characteristic function of the temperature process is given by:

$$\varphi_{T_t}^\theta(u) \quad = \quad exp(iuC_1(t,\alpha)) \left(\sum_{k=0}^{+\infty} C_2(u,\theta,k) C_3(u,\theta,k) \right) m_J(u,\theta)$$

(14)

where:

$$
\begin{aligned}
J_l^\theta(u) &= \int_0^{+\infty} e^{I_l^\theta(u,x)} e^{-(\lambda_l-\theta)x}\, dx, \quad l=1,2 \\
J_3^\theta(u,k) &= D(k,k,2k)\sum_{l=0}^{+\infty} C(k,2k,l) G_1^\theta(u,t,2k+l)\lambda_2^l \\
J_4^\theta(u,k) &= D(k+1,k,2k+1)\sum_{l=0}^{+\infty} C(k+1,2k+1,l) G_2^\theta(u,t,2k+l+1)\lambda_2^l \\
m_J(u,\theta) &= \sum_{k=0}^{+\infty} J_3^\theta(u,k) P_\theta(N_t=2k) + \sum_{k=0}^{+\infty} J_4^\theta(u,k) P_\theta(N_t=2k+1) \\
G_j^\theta(u,m) &= \int_0^t e^{I_j^\theta(u,t-z)} z^{m-1} e^{-(\lambda_2-\theta)z}\, dz, \; j=1,2 \\
C_2(u,\theta,k) &= (\lambda_{12}-\theta)^k(\lambda_{21}-\theta)^k (J_1^\theta(u))^k (J_2^\theta(u))^k \\
C_3(u,\theta,k) &= P_\theta(N_t=2k) + (\lambda_{12}-\theta) J_1^\theta(u) P_\theta(N_t=2k+1)
\end{aligned}
$$

$$
I_j^\theta(u,x) = \int_0^x l_{R^j}(-iu\mu_1^j\sigma e^{-\alpha(t-s)} + \mu_1^j\theta + \frac{1}{2}(-iu\sigma e^{-\alpha(t-s)}+\theta)^2)\, ds - l_{R^j}(\mu_1^j\theta + \frac{1}{2}\theta^2))x
$$

Proof. Notice that from the change of probability defined by the Esscher transform:

$$
\varphi^\theta_{V_t^j}(u) = E(e^{iuV_t^j} e^{\theta V_t^j - l_{V_t^j}(\theta)}) = \frac{\varphi_{V_t^j}(u-i\theta)}{\varphi_{V_t^j}(-i\theta)}
$$

and $l^\theta_{V^j}(u) = l_{V^j}(u+\theta) - l_{V^j}(\theta)$.
Theorem 5 is applied with $\xi_t^l = V_t^l, f(s) = u\sigma e^{-\alpha(t-s)}$ to get equation (14).
Moreover:

$$
\begin{aligned}
I_j^\theta(u,x) &= \int_0^x l^\theta_{V^j}(-iu\sigma e^{-\alpha(t-s)})\, ds = \int_0^x l_{V^j}(-iu\sigma e^{-\alpha(t-s)}+\theta)\, ds - l_{V^j}(\theta)x \\
&= \int_0^x l_{R^j}(-iu\mu_1^j\sigma e^{-\alpha(t-s)} + \mu_1^j\theta + \frac{1}{2}(-iu\sigma e^{-\alpha(t-s)}+\theta)^2)\, ds \\
&- l_{R^j}(\mu_1^j\theta + \frac{1}{2}\theta^2)x
\end{aligned}
$$

The m.g.f. of the number of regime changes on $[0,t)$ under the Esscher transform becomes:

$$
\begin{aligned}
M_{N_t}(\theta) &= \sum_{k=0}^{+\infty} e^{\theta k} P(N_t=k) = \sum_{k=0}^{+\infty} e^{2k\theta} P(N_t=2k) + \sum_{k=0}^{+\infty} e^{\theta(2k+1)} P(N_t=2k+1) \\
&= \sum_{k=0}^{+\infty} e^{2k\theta}\left(P(N_t=2k) + e^\theta P(N_t=2k+1)\right) \\
&= \sum_{k=0}^{+\infty} e^{2k\theta}\left(\frac{\lambda_1^k}{\lambda_2^k\Gamma(2k)} B_1(t,k) + \frac{\lambda_1^{k+1}e^\theta}{\lambda_2^{k+1}\Gamma(2k+1)} B_2(t,k)\right) \\
&= \sum_{k=0}^{+\infty} e^{2k\theta} D(k,-k,2k)\left(B_1(t,k) + \frac{\lambda_{12}e^\theta}{\lambda_{21}(2k+1)} B_2(t,k)\right)
\end{aligned}
$$

Furthermore, under the measure $\mathcal{Q}^\theta$ the times between two consecutive regime changes $v_j \sim exp(\lambda_j - \theta)$ for $\theta < \lambda_j$, independently each other.
Consequently, the p.d.f. and the c.d.f. of τ_k under $\mathcal{Q}^\theta$, denoted respectively $f_{\tau_k}(x,\theta)$ and $F_{\tau_k}(x,\theta)$ are given by:

$$f_{\tau_k}(x,\theta) \quad = \quad e^{\theta x} f_{\tau_k}(x), \; x \geq 0 \; \theta < \lambda_{21} \; k \in \mathbb{N}$$

□

The result below specifies the value of θ under the Esscher transform to obtain an EMM. For practical proposes with restrain the EMM condition the case $s = 0$ and any $t \geq 0$. Hence the condition reads $E[\tilde{T}_t] = \tilde{T}_0$. The general case follows in a similar way.
We first introduce the following quantities:

$$\begin{aligned}
C_4(u,\theta) &= \sum_{k=0}^{+\infty} C_2(u,\theta,k) C_3(u,\theta,k) \\
C_5(u,\theta) &= \sum_{k=0}^{+\infty} J_3^\theta(u,k) P_\theta(N_t = 2k) + \sum_{k=0}^{+\infty} J_4^\theta(u,k) P_\theta(N_t = 2k+1) \\
C_6(u,\theta) &= 1 - ik\alpha \left[\frac{L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)} + \frac{L_1^2(\theta,\alpha,\sigma)}{(\lambda_{21}-\alpha-\theta)} \right] \\
C_7(u,\theta) &= -i\alpha \left[\frac{(k+1) L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)} + \frac{L_1^2(\theta,\alpha,\sigma)}{(\lambda_{21}-\alpha-\theta)} \right] \\
L_1^j(\theta,\alpha,\sigma) &= (\alpha\sigma)^{-1} (\theta + \mu_1^j) e^{-(\alpha+2r)t} \frac{dl_{R^j}}{du} (\mu_1^j \theta + \frac{1}{2}\theta^2)
\end{aligned}$$

and the functions $I_j^\theta(u,x), J_l^\theta(u), G_j^\theta(u,t,m), J_3(u,t,k,\theta)$ and $J_4(u,t,k,\theta)$ are defined as in Theorem 7.

Theorem 8. Let $(T_t)_{t\geq 0}$ be the temperature process defined by equations (1), (2) and (12) starting at regime one. Then, the Esscher measure $\mathcal{Q}^\theta$ is an EMM if for any $t > 0$ the parameter θ verifies under the conditions $\theta \leq min(\lambda_l - \alpha, 0), \lambda_{12} < \lambda_{21}$ $l = 1,2$:

$$R(t,\theta) = \sigma^{-1} e^{(\alpha+r)t} (T_0 - e^{-rt} C_1(t,\alpha)) \tag{15}$$

where:

$$R(t,\theta) \quad = \quad -i \left[\frac{dC_4(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0} C_5(0,\theta,k) + \frac{dC_5(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0} \right]$$

The expressions $\frac{dC_4(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}$, $\frac{dC_5(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}$ and C_5 are given respectively by equations (16), (17) and (18).

Proof. From Lemma 2 the discounted temperature process $(\tilde{T}_t)_{t\geq 0}$ verifies:

$$\tilde{T}_t = \tilde{C}_1(t,\alpha) + \sigma e^{-\alpha t} \tilde{W}_t$$

Then the EMM condition for $s = 0$ translates into:

$$\begin{aligned}
E_\theta(\tilde{T}_t) &= \tilde{T}_0 \\
& \Leftrightarrow E_\theta[e^{-rt} T_t] = E_\theta[e^{-rt} C_1(t,\alpha) + \sigma e^{-(\alpha+r)t} W_t] \\
&= e^{-rt} C_1(t,\alpha) + \sigma e^{-(\alpha+r)t} E_\theta[W_t] = T_0
\end{aligned}$$

or equivalently the parameter θ solves:

$$E_\theta[W_t] \quad = \quad \sigma^{-1}e^{(\alpha+r)t}(T_0 - e^{-rt}C_1(t,\alpha))$$

We call Theorem 7 to compute the moment via its characteristic function. To this end notice that

$$W_t = \sigma^{-1}e^{(\alpha+r)t}(T_t - e^{-rt}C_1(t,\alpha))$$

Consequently:

$$\begin{aligned}
\varphi^\theta_{W_t}(u) &= exp(-iu\sigma^{-1}e^{-rt})\varphi\theta_{T_t}(u\sigma^{-1}e^{-rt}) \\
&= exp(-iu\sigma^{-1}e^{-rt})exp(iu\sigma^{-1}e^{-rt})C_4(u\sigma^{-1}e^{-rt},\theta)C_5(u\sigma^{1}e^{-rt},\theta) \\
&= C_4(u\sigma^{1}e^{-rt},\theta)C_5(u\sigma^{1}e^{-rt},\theta)
\end{aligned}$$

where:

$$\begin{aligned}
C_4(u\sigma^1e^{-rt},\theta) &= \sum_{k=0}^{+\infty} C_2(u\sigma^1e^{-rt},\theta,k)C_3(u\sigma^1e^{-rt},\theta,k) \\
C_5(u\sigma^1e^{-rt},\theta) &= \sum_{k=0}^{+\infty} J^\theta_3(u\sigma^1e^{-rt},k)P_\theta(N_t=2k) + \sum_{k=0}^{+\infty} J^\theta_4(u\sigma^1e^{-rt},k)P_\theta(N_t=2k+1)
\end{aligned}$$

Notice that $C_2(0,\theta,k)=1$ and $C_3(0,\theta,k)=P_\theta(N_t=2k)+P_\theta(N_t=2k+1)$.
The computation of the derivatives of the intermediate functions $I^\theta_j(u,x), J^\theta_l(u), G^\theta_j(u,t,m)$, $J_3(u,t,k,\theta), J_4(u,t,k,\theta), C_2(u,\theta)$ and $C_3(u,\theta)$ is straightforward. They are left to the Appendix.
Then,

$$\begin{aligned}
\frac{dC_4}{du}(u\sigma^{-1}e^{-rt},\theta) &= \sum_{k=0}^{+\infty}\left[\frac{dC_2}{du}(u\sigma^{-1}e^{-rt},\theta,k)C_3(u\sigma^{-1}e^{-rt},\theta,k)\right. \\
&+ \left.\frac{dC_3}{du}(u\sigma^{-1}e^{-rt},\theta,k)C_2(u\sigma^{-1}e^{-rt},\theta,k)\right] \\
\frac{dC_4}{du}(u\sigma^{-1}e^{-rt},\theta)|_{u=0} &= \sigma^{-1}e^{-rt}\sum_{k=0}^{+\infty}\left[\frac{dC_2}{du}(u\sigma^{-1}e^{-rt},\theta,k)|_{u=0}C_3(0,\theta,k)\right. \\
&+ \left.\frac{dC_3}{du}(u\sigma^{-1}e^{-rt},\theta,k)|_{u=0}\right] \\
&= \sum_{k=0}^{+\infty}\left[-ik\alpha\left[\frac{L^1_1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)}+\frac{L^2_1(\theta,\alpha,\sigma)}{(\lambda_{21}-\alpha-\theta)}\right][P_\theta(N_t=2k)+P_\theta(N_t=2k+1)]\right. \\
&+ \left.[P_\theta(N_t=2k) - i\frac{\alpha L^1_1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)}P_\theta(N_t=2k+1)]\right] \\
&= \sum_{k=0}^{+\infty} C_6(u,\theta)P_\theta(N_t=2k)+C_7(u,\theta)P_\theta(N_t=2k+1)]
\end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{dC_5}{du}(u\sigma^{-1}e^{-rt},\theta) &= \sum_{k=0}^{+\infty}\frac{dJ_3^\theta}{du}(u\sigma^{-1}e^{-rt},k)P_\theta(N_t=2k)\\ &+ \sum_{k=0}^{+\infty}\frac{dJ_4^\theta}{du}(u\sigma^{-1}e^{-rt},k)P_\theta(N_t=2k+1)\\ \frac{dC_5}{du}(u\sigma^{1}e^{-rt},\theta,k)|_{u=0} &= \sum_{k=0}^{+\infty}\frac{dJ_3^\theta}{du}(u\sigma^{-1}e^{-rt},k)|_{u=0}P_\theta(N_t=2k)\\ &+ \sum_{k=0}^{+\infty}\frac{dJ_4^\theta}{du}(u\sigma^{-1}e^{-rt},k)|_{u=0}P_\theta(N_t=2k+1)\end{aligned}$$

(17)

Notice that:

$$\begin{aligned}G_j^\theta(0,m) &= \int_0^t z^{m-1}e^{-(\lambda_2-\theta)z}\,dz = (\lambda_2-\theta)^{-1}\Gamma(m,(\lambda_2-\theta)t)\\ J_3^\theta(0,k) &= D(k,k,2k)\sum_{l=0}^{+\infty}C(k,2k,l)G_j^\theta(0,2k+l)\lambda_2^l\\ J_4^\theta(0,k) &= D(k+1,k,2k+1)\sum_{l=0}^{+\infty}C(k+1,2k+1,l)G_j^\theta(0,2k+l+1)\lambda_2^l\\ C_2(0,\theta,k) &= 1\\ C_3(0,\theta,k) &= P_\theta(N_t=2k)+P_\theta(N_t=2k+1)\end{aligned}$$

Leading to:

$$\begin{aligned}C_4(0,\theta) &= \sum_{k=0}^{+\infty}C_2(0,\theta,k)C_3(0,\theta,k) = \sum_{k=0}^{+\infty}C_3(0,\theta,k)\\ &= \sum_{k=0}^{+\infty}(P_\theta(N_t=2k)+P_\theta(N_t=2k+1)) = 1\\ C_5(0,\theta) &= \sum_{k=0}^{+\infty}J_3^\theta(0,k)P_\theta(N_t=2k)+\sum_{k=0}^{+\infty}J_4^\theta(0,k)P_\theta(N_t=2k+1)\end{aligned}$$

(18)

$$\begin{aligned}\frac{d\varphi_{W_t^\theta}(u\sigma^{-1}e^{-rt})}{du}|_{u=0} &= \frac{dC_4(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}C_5(0,\theta,k)+\frac{dC_5(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}C_4(0,\theta,k)\\ &= \frac{dC_4(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}C_5(0,\theta,k)+\frac{dC_5(u\sigma^{-1}e^{-rt},\theta,k)}{du}|_{u=0}\end{aligned}$$

□

4. Case of Gamma and Inverse Gaussian subordinators

We analyze in details the calculation of the characteristic function and the Esscher parameter in the case of the model (1) with Gamma and Inverse Gaussian subordinators.

As no closed-form formulas exist, numerical approximations are required. It involves truncations in the probability distribution of the changes of regimes on $[0, t]$, as well as in other intermediate related series. Also, numerical integration, following trapezoidal rule, is necessary in repeated occasions. Generally speaking the order of calculation followed to obtain the characteristic function under the Esscher transform is the following:

$$D, C, \Gamma, B_j, M_\theta(k), P_\theta(k), l^\theta_{R^j}, I^\theta_j, J^\theta_l, G^\theta_l, J^\theta_l(u,k), C_2, C_3, \varphi^\theta_{T_t}$$

Example 9. *Gamma subordinator* Consider the subordinators $(R^j_t)_{t\geq 0}$ are Gamma processes with parameters $a_j > 0, b_j > 0, j = 1, 2$ with respective characteristic function and Laplace exponent:

$$\begin{aligned} \varphi_{R^j}(u) &= \left(1 - \frac{iu}{b_j}\right)^{-a_j t}, \ a_j > 0, b_j > 0 \\ l_{R^j}(u) &= -a_j \log\left(1 - \frac{u}{b_j}\right), \ u < b_j \end{aligned}$$

and

$$\begin{aligned} \varphi^\theta_{R^j}(u) &= \left(1 - \frac{ih_j(u,s,\theta)}{b_j}\right)^{-a_j t}, \ a_j > 0, b_j > 0 \\ l^\theta_{R^j}(u) &= -a_j \log\left(1 - \frac{h_j(u,s,\theta)}{b_j}\right), \ u < b_j \end{aligned}$$

where:

$$h_j(u,s,\theta) = u\mu^j_1 \sigma e^{-\alpha t} e^{\alpha s} + \mu^j_1\theta + \frac{1}{2}(u\sigma e^{-\alpha t} e^{\alpha s} + \theta)^2, \ j = 1, 2.$$

Therefore:

$$\begin{aligned} I^\theta_j(u,x) &= \int_0^x l_{R^j} h_j(u,s,\theta)\, ds - l_{R^j}(\mu^j_1\theta + \frac{1}{2}\theta^2)x \\ &= -a_j \left[\int_0^x \log\left(1 - \frac{h_j(u,s,\theta)}{b_j}\right) ds - log\left(1 - \frac{(\mu^j_1\theta + \frac{1}{2}\theta^2)}{b_j}\right) x\right] \\ J^\theta_l(u) &= \int_0^{+\infty} e^{I^\theta_l(u,x)} e^{-(\lambda_l - \theta)x}\, dx \\ &= \left(1 - \frac{\mu^j_1\theta + \frac{1}{2}\theta^2}{b_j}\right)^{-a_j x} \int_0^{+\infty} exp(-a_j \int_0^x \log\left(1 - \frac{h_j(u,s,\theta)}{b_j}\right) ds - (\lambda_l - \theta)x)\, dx \\ G^\theta_j(u,m) &= \int_0^t e^{I^\theta_j(u,t-z)} z^{m-1} e^{-(\lambda_2 - \theta)z}\, dz, \ j = 1, 2 \\ &= \left(1 - \frac{\mu^j_1\theta + \frac{1}{2}\theta^2}{b_j}\right)^{-a_j t} \\ &\quad \int_0^t \left(1 - \frac{\mu^j_1\theta + \frac{1}{2}\theta^2}{b_j}\right)^{a_j z} \\ &\quad \exp(-a_j \int_0^{t-z} \log\left(1 - \frac{h_j(u,s,\theta)}{b_j}\right) ds) z^{m-1} e^{-(\lambda_2 - \theta)z}\, dz \end{aligned}$$

We compute the Gerber-Shiu parameter θ from the martingale condition given by equation (15). First,

$$\begin{aligned}
\frac{dl_{R^j}}{du}(\mu_1^j\theta+\frac{1}{2}\theta^2) &= \frac{a_j}{b_j-\mu_1^j\theta-\frac{1}{2}\theta^2}\\
L_1^j(\theta,\alpha,\sigma) &= (\alpha\sigma)^{-1}(\theta+\mu_1^j)e^{-(\alpha+2r)t}\frac{dl_{R^j}}{du}(\mu_1^j\theta+\frac{1}{2}\theta^2)\\
&= \frac{a_j(\alpha\sigma)^{-1}(\theta+\mu_1^j)e^{-(\alpha+2r)t}}{b_j-\mu_1^j\theta-\frac{1}{2}\theta^2}
\end{aligned}$$

The previous calculations allow to find C_6, C_7, C_4 and C_5 and its derivatives leading to $R(t,\theta)$ in equation (15).

Example 10. *Switching Inverse Gaussian subordinator*

$$\begin{aligned}
\varphi_{R^j}(u) &= exp(-a_j\sqrt{-2iu+b_j^2}-b_j)\\
l_{R^j}(u) &= -a_j\sqrt{-2u+b_j^2}-b_j
\end{aligned}$$

and

$$\begin{aligned}
\varphi^\theta_{R^j}(u) &= exp(-a_j\sqrt{-2ih_j(u,s,\theta)+b_j^2}-b_j)\\
l^\theta_{R^j}(u) &= -a_j\sqrt{-2h_j(u,s,\theta)+b_j^2}-b_j
\end{aligned}$$

$$\begin{aligned}
I_j^\theta(u,x) &= \int_0^x l_{R^j}(h_j(u,s,\theta))\,ds - l_{R^j}(\mu_1^j\theta+\frac{1}{2}\theta^2)x\\
&= -a_j\left[\int_0^x\sqrt{-2h_j(u,s,\theta)+b_j^2}\,ds-(b_j+\sqrt{-2(\mu_1^j\theta+\frac{1}{2}\theta^2)+b_j^2})x\right]\\
J_l^\theta(u) &= \int_0^{+\infty} exp(-a_j(\int_0^x\sqrt{-2h_j(u,s,\theta)+b_j^2}\,ds)exp(-(b_j+\sqrt{-2(\mu_1^j\theta+\frac{1}{2}\theta^2)+b_j^2}\lambda_l-\theta)x)\,dx\\
&=\\
G_j^\theta(u,m) &= \int_0^t e^{I_j^\theta(u,t-z)}z^{m-1}e^{-(\lambda_2-\theta)z}\,dz\\
&= \int_0^t exp(-a_j\int_0^{t-z}\sqrt{-2h_j(u,s,\theta)+b_j^2}\,ds\\
&\quad -a_j(b_j+\sqrt{-2(\mu_1^j\theta+\frac{1}{2}\theta^2)+b_j^2})(t-z))z^{m-1}e^{-(\lambda_2-\theta)z}\,dz\\
&= \left(b_j+\sqrt{-2(\mu_1^j\theta+\frac{1}{2}\theta^2)+b_j^2}\right)t\\
&\quad \int_0^t exp(-a_j\left[\int_0^{t-z}\sqrt{-2h_j(u,s,\theta)+b_j^2}\,ds+(b_j+\sqrt{-2(\mu_1^j\theta+\frac{1}{2}\theta^2)+b_j^2})z\right])z^{m-1}e^{-(\lambda_2-\theta)z}\,dz
\end{aligned}$$

5. Conclusions

A switching mean-reverting model for temperatures when the latter oscillates between two stochastic differential equations with time-changed Levy noises has

been investigated. The characteristic function and Esscher parameter to produce an Equivalent Martingale Measure have been found via an approximate closed-form expression. Calculations require some rather complex but feasible numerical approximations involving numerical calculations of double integrals, truncated series and root solving.
This theoretical framework paves the way to price weather financial contracts based on temperature indices such as cumulative average temperatures, cooling and heating days.

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7. Appendix

To compute the derivatives of the characteristic function we first find the derivatives of the intermediate functions $I_j^\theta(u,x)$, $J_l^\theta(u)$, $G_j^\theta(u,t,m)$, $J_3^\theta(u,t,k,\theta)$, $J_4^\theta(u,t,k,\theta)$, $C_2(u,\theta,k)$ and $C_3(u,\theta,k)$.
To this end we denote by:

$$g_j(u\sigma^{-1}e^{-rt},s,\theta) = -iu\mu_1^j e^{-(\alpha+r)t}e^{\alpha s} + \mu_1^j\theta + \frac{1}{2}(-iue^{-(\alpha+r)t}e^{\alpha s}+\theta)^2,\ j=1,2.$$

we have:

$$\frac{dg_j(u\sigma^{-1}e^{-rt},s,\theta)}{du} = -i\mu_1^j e^{-(\alpha+r)t}e^{\alpha s} - ie^{-(\alpha+r)t}e^{\alpha s}(-iue^{-(\alpha+r)t}e^{\alpha s}+\theta),\ j=1,2.$$

Then:

$$\begin{aligned}
\frac{dI_j^\theta(u\sigma^{-1}e^{-rt},x)}{du}|_{u=0} &= \sigma^{-1}e^{-rt}\int_0^x \frac{dl_{R^j}}{du}(g_j(u\sigma^{-1}e^{-rt},s,\theta))|_{u=0}\frac{dg_j(u\sigma^{-1}e^{-rt},s,\theta)}{du}|_{u=0}\ ds \\
&= -i\sigma^{-1}(\theta+\mu_1^j)e^{-(\alpha+2r)t}\int_0^x \frac{dl_{R^j}}{du}(\mu_1^j\theta+\frac{1}{2}\theta^2)e^{\alpha s}\ ds \\
&= -i\sigma^{-1}(\theta+\mu_1^j)e^{-(\alpha+2r)t}\frac{dl_{R^j}}{du}(\mu_1^j\theta+\frac{1}{2}\theta^2)\int_0^x e^{\alpha s}\ ds \\
&= -iL_1(\theta,\alpha,\sigma)(e^{\alpha x}-1)
\end{aligned}$$

Hence:

$$\frac{dG_j^\theta(u\sigma^{-1}e^{-rt},t,m)}{du} = \int_0^t \frac{dI_j^\theta(u\sigma^{-1}e^{-rt},t-z)}{du}e^{I_j^\theta(u\sigma^{-1}e^{-rt},t-z)}z^{m-1}e^{-(\lambda_{21}-\theta)z}\ dz$$

$$\frac{dG_j^\theta(u,t,m)}{du}|_{u=0} = -iL_1(\theta,\alpha,\sigma)\int_0^t e^{I_j^\theta(0,t-z)}(e^{\alpha(t-z)}-1)z^{m-1}e^{-(\lambda_{21}-\theta)z}\ dz$$

$$= -iL_1^j(\theta,\alpha,\sigma,t)\left[\int_0^t e^{I_j^\theta(0,t-z)}z^{m-1}e^{-(\lambda_{21}-\theta)z+\alpha(t-z)}\ dz - \int_0^t z^{m-1}e^{-(\lambda_{21}-\theta)z}\ dz\right]$$

$$= -iL_1^j(\theta,\alpha,\sigma)\left[e^{\alpha t}\int_0^t z^{m-1}e^{-(\lambda_{21}-\theta+\alpha)z}\ dz - \int_0^t z^{m-1}e^{-(\lambda_{21}-\theta)z}\ dz\right]$$

$$\begin{aligned}
&= -iL_1^j(\theta,\alpha,\sigma)\left[e^{\alpha t}\int_0^t z^{m-1}exp(-(\lambda_{21}-\theta+\alpha)z)\,dz - \int_0^t z^{m-1}exp(-(\lambda_{l2}-\theta)z)\,dz\right]\\
&= -iL_1^j(\theta,\alpha,\sigma)\left[e^{\alpha t}\int_0^t z^{m-1}exp(-(\lambda_{2}-\theta+\alpha)z)\,dz - \int_0^t z^{m-1}exp(-(\lambda_{l}-\theta)z)\,dz\right]\\
&= -iL_1^j(\theta,\alpha,\sigma)\left[e^{\alpha t}\int_0^t z^{m-1}exp(-(\alpha+\lambda_{21}-\theta)z)\,dz - \int_0^t z^{m-1}exp(-(\lambda_{21}-\theta)z)\,dz\right]\\
&= -iL_1^j(\theta,\alpha,\sigma)\left[e^{\alpha t}(\alpha+\lambda_{21}-\theta)^{-1}\Gamma(m,\alpha+\lambda_{21}-\theta)-(\lambda_{21}-\theta)^{-1}\Gamma(m,\lambda_{21}-\theta)\right]
\end{aligned}$$

Moreover for $j=1,2$:

$$\begin{aligned}
\frac{dJ_l^\theta(u\sigma^{-1}e^{-rt})}{du} &= \int_0^{+\infty}\frac{dI_l^\theta(u\sigma^{-1}e^{-rt},x)}{du}e^{I_l^\theta(u\sigma^{-1}e^{-rt},x)}e^{-(\lambda_l-\theta)x}\,dx\\
\frac{dJ_l^\theta(u\sigma^{-1}e^{-rt})}{du}|_{u=0} &= \int_0^{+\infty}\frac{dI_l^\theta(u\sigma^{-1}e^{-rt},x)}{du}|_{u=0}e^{I_l^\theta(0,x)}e^{-(\lambda_l-\theta)x}\,dx\\
&= -iL_1^j(\theta,\alpha,\sigma)\int_0^{+\infty}(e^{\alpha x}-1)e^{-(\lambda_l-\theta)x}\,dx\\
&= -iL_1^j(\theta,\alpha,\sigma)\int_0^{+\infty}(e^{\alpha x}-1)e^{-(\lambda_l-\theta)x}\,dx\\
&= -iL_1^j(\theta,\alpha,\sigma)\left[\int_0^{+\infty}e^{\alpha x}e^{-(\lambda_l-\theta)x}\,dx-\int_0^{+\infty}e^{-(\lambda_l-\theta)x}\,dx\right]\\
&= -iL_1^j(\theta,\alpha,\sigma)\left[\frac{1}{\lambda_l-\alpha-\theta}-\frac{1}{\lambda_l-\theta}\right]\\
&= -\frac{i\alpha L_1^j(\theta,\alpha,\sigma)}{(\lambda_l-\theta)(\lambda_l-\alpha-\theta)}
\end{aligned}$$

(19)

under the condition $\theta \le max(\lambda_l-\alpha,0)$.
On the other hand:

$$\begin{aligned}
\frac{d(J_1^\theta(u\sigma^{-1}e^{-rt}))^k(J_2^\theta(u\sigma^{-1}e^{-rt}))^k}{du} &= \frac{d(J_1^\theta(u\sigma^{-1}e^{-rt}))^k}{du}(J_2^\theta(u\sigma^{-1}e^{-rt}))^k\\
&+ \frac{d(J_2^\theta(u\sigma^{-1}e^{-rt}))^k}{du}(J_1^\theta(u\sigma^{-1}e^{-rt}))^k\\
&= k(J_1^\theta(u\sigma^{-1}e^{-rt}))^{k-1}(J_2^\theta(u\sigma^{-1}e^{-rt}))^k\frac{dJ_1^\theta(u\sigma^{-1}e^{-rt})}{du}\\
&+ k(J_2^\theta(u\sigma^{-1}e^{-rt}))^{k-1}(J_1^\theta(u\sigma^{-1}e^{-rt}\sigma^{-1}e^{-rt}))^k\frac{dJ_2^\theta(u\sigma^{-1}e^{-rt})}{du}
\end{aligned}$$

Hence:

$$\begin{aligned}
\frac{d(J_1^\theta(u\sigma^{-1}e^{-rt}))^k(J_2^\theta(u\sigma^{-1}e^{-rt}))^k}{du}|_{u=0} &= \sigma^{-1}e^{-rt}k(J_1^\theta(0))^{k-1}(J_2^\theta(0))^k\frac{dJ_1^\theta(u\sigma^{-1}e^{-rt})}{du}|_{u=0}\\
&+ \sigma^{-1}e^{-rt}k(J_2^\theta(0))^{k-1}(J_1^\theta(0))^k\frac{dJ_2^\theta(u\sigma^{-1}e^{-rt})}{du}|_{u=0}\\
&= k(J_1^\theta(0))^{k-1}(J_2^\theta(0))^{k-1}\left[\frac{dJ_1^\theta(u\sigma^{-1}e^{-rt})}{du}|_{u=0}J_2^\theta(0)\right.\\
&+ \left.\frac{dJ_2^\theta(u\sigma^{-1}e^{-rt})}{du}|_{u=0}J_1^\theta(0)\right]
\end{aligned}$$

But, taking into account that:

$$\begin{aligned} I_l^{\theta}(0,x) &= \int_0^x l_{R^j}(\mu_1^j\theta + \frac{1}{2}\theta^2)\,ds - l_{R^j}(\mu_1^j\theta + \frac{1}{2}\theta^2))x = 0 \\ J_l^{\theta}(0) &= \int_0^{+\infty} e^{-(\lambda_l-\theta)x}\,dx = \frac{1}{\lambda_l-\theta} \end{aligned}$$

we have:

$$\begin{aligned} \frac{d(J_1^{\theta}(u))^k(J_2^{\theta}(u))^k}{du}|_{u=0} &= \frac{k}{(\lambda_{12}-\theta)^{k-1}(\lambda_{21}-\theta)^{k-1}} \\ &\quad \left[\frac{-i\alpha L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\theta)(\lambda_{12}-\alpha-\theta)(\lambda_{21}-\theta)} - \frac{i\alpha L_1^2(\theta,\alpha,\sigma,t)}{(\lambda_{21}-\theta)(\lambda_{21}-\alpha-\theta)(\lambda_{12}-\theta)}\right] \\ &= -i\frac{k\alpha}{(\lambda_{12}-\theta)^k(\lambda_{21}-\theta)^k}\left[\frac{L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)} + \frac{L_1^2(\theta,\alpha,\sigma)}{(\lambda_{21}-\alpha-\theta)}\right] \end{aligned} \tag{20}$$

Moreover:

$$\begin{aligned} \frac{dJ_3(u\sigma^{-1}e^{-rt}),k,\theta)}{du}|_{u=0} &= D(k,k,2k)\sum_{l=0}^{+\infty} C(k,2k,l)\frac{dG_1^{\theta}}{du}(u\sigma^{-1}e^{-rt},2k+l)|_{u=0}\lambda_2^l \\ \frac{dJ_4(u\sigma^{-1}e^{-rt},k,\theta)}{du}|_{u=0} &= D(k+1,k,2k+1)\sum_{l=0}^{+\infty} C(k+1,2k+1,l)\frac{dG_2^{\theta}}{du}(u\sigma^{-1}e^{-rt},2k+l+1)|_{u=0}\lambda_2^l \end{aligned} \tag{21}$$

Then, taking into account equations (19),(21) and (20) we have:

$$\begin{aligned} \frac{dC_2}{du}(u\sigma^{-1}e^{-rt},\theta,k) &= (\lambda_{12}-\theta)^k(\lambda_{21}-\theta)^k\frac{d(J_1^{\theta}(u\sigma^{-1}e^{-rt}))^k(J_2^{\theta}(u\sigma^{-1}e^{-rt}))^k}{du} \\ \frac{dC_2}{du}(u\sigma^{-1}e^{-rt},\theta,k)|_{u=0} &= -ik\alpha\left[\frac{L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)} + \frac{L_1^2(\theta,\alpha,\sigma)}{(\lambda_{21}-\alpha-\theta)}\right] \end{aligned}$$

and

$$\begin{aligned} \frac{dC_3}{du}(u\sigma^{-1}e^{-rt},\theta,k) &= P_{\theta}(N_t=2k) + (\lambda_{12}-\theta)\frac{dJ_1^{\theta}(u)}{du}P_{\theta}(N_t=2k+1) \\ \frac{dC_3}{du}(u\sigma^{-1}e^{-rt},\theta,k)|_{u=0} &= P_{\theta}(N_t=2k) + (\lambda_{12}-\theta)\frac{dJ_1^{\theta}(u\sigma^{-1}e^{-rt})}{du}|_{u=0}P_{\theta}(N_t=2k+1) \\ &= P_{\theta}(N_t=2k) - i\frac{\alpha L_1^1(\theta,\alpha,\sigma)}{(\lambda_{12}-\alpha-\theta)}P_{\theta}(N_t=2k+1) \end{aligned}$$

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