PRICING EXCHANGE OPTIONS UNDER STOCHASTIC CORRELATION

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[1]

Abstract. In this paper we study the pricing of exchange options when underlying assets have stochastic volatility and stochastic correlation. An approximated closed-form formula based on a Taylor expansion of the conditional Margrabe price is proposed. The problem is illustrated within the framework of the exchange between two different types of oil commodities.

Keywords. exchange options, stochastic correlation, Taylor expansion, Ornstein–Uhlenbek process.

1. Introduction

In this paper we study the pricing of exchange options when the underlying assets have stochastic volatility and correlation. Its main contribution is the proposal of an approximated closed-form methodology for the price of the contract, providing an accurate valuation under a more realistic model within a reasonable computing time. The results provide some comparison with the classic Margrabe setting in terms of sensitivities with respect ot different risk factors.
The pricing of exchange contracts has been first considered in [10] under a standard bivariate Black-Scholes model, where a closed-form formula for the pricing is provided. Unfortunately, once outside the classic setting there is not closed-form expression to valuate the price. In [6] the pricing of single-asset option contract under stochastic volatility is studied. The idea is extended in [8, 17, 18] to exchanges in the case of stochastic volatilities while the pricing in models with stochastic correlation and constant volatilities is covered in [1] for a related spread option contract. Using some approximations and ad-hoc methods the pricing have been considered in [3, 4] in the case of a jump-diffusion model, and in [2] for the pricing of stochastic interest rates.
An exchange between two assets impacted with climate change, e.g. companies with positive and negative exposure to environmental risk, allows to hedge against catastrophe risk. For the pricing of exchange options in catastrophe risk management, see for example [20]. Exchange option contracts can be used within the framework of real options analysis by considering the exchange between two different projects that could possibly undertaken, one sustainable and the other not, whose future returns are subject to uncertainty, see [16]. Exchanges in the context of credit risk have been studied in [5] and [15].

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In the current paper we specifically consider the pricing of the exchange under a model whose volatilities follow correlated Ornstein-Ulenbeck models and a mean-reverting stochastic dynamic for the correlation. In [19] a model with stochastic covariance is taken into account.
In [7] a pricing method based on a Taylor expansion has been considered for European single-asset derivatives, whereas the same approach but for spread options have been studied in [9, 11, 13, 12, 19] under different dynamic models and contracts.
The organization of the paper is the following:
In section 2 the model is introduced, the approximated pricing formula is discussed and the first and second order moments of the squered volatilities and correlation between assets are computed. Their proofs are deferred to the appendix. In section 3 we discuss the numerical results for the pricing of exchange options between two different types of oil commodities. Finally, a discussion of the results and recommendations is presented in the conclusion section.

2. Pricing exchange options in models with stochastic correlation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. We denote by $\mathcal{Q}$ a risk-neutral equivalent martingale measure(EMM) and $E_{\mathcal{Q}}$ the expected value with respect to the measure $\mathcal{Q}$.
For a stochastic process $(X_t)_{t\geq 0}$, the integrated model associated with it is denoted by $(X^+_t)_{t\geq 0}$ and defined as:

$$X^+_t = \int_0^t X_s \, ds$$

The functions $f_X(x)$ and $f_{X/Y}(x/y)$ are respectively the probability density function (p.d.f.) of the random vector X and the conditional p.d.f. of X on another random vector Y.
A two-dimensional adapted stochastic process $(S_t)_{t\geq 0} = (S^{(1)}_t, S^{(2)}_t)_{t\geq 0}$, where their components are prices of certain underlying assets, is defined on the filtered probability space above.
We assume that the process of prices has a dynamic under $\mathcal{Q}$ given by:

$$\begin{aligned} dS^{(1)}_t &= r\, S^{(1)}_t\, dt + \sigma^{(1)}_t\, S^{(1)}_t\, dZ^{(1)}_t \\ dS^{(2)}_t &= r\, S^{(2)}_t\, dt + \sigma^{(2)}_t \sqrt{1-\rho_t^2}\, S^{(2)}_t\, dZ^{(2)}_t + \sigma^{(2)}_t\, \rho_t\, S^{(2)}_t\, dZ^{(1)}_t \end{aligned}$$

where $(\sigma_t)_{t\geq 0} = (\sigma^{(1)}_t, \sigma^{(2)}_t)_{t\geq 0}$ is the bivariate volatility process, $V_t = (V^{(1)}_t, V^{(2)}_t)_{t\geq 0}$, with $V^{(j)}_t = (\sigma^{(j)}_t)^2, j = 1, 2$ is the process of squared volatilities of the underlying assets and ρ_t is the linear correlation coefficient between the assets at time t, whose dynamic is given by:

$$d\rho_t = \bar{\gamma}\,(\bar{\Gamma}_L - \rho_t)\, dt + \bar{\alpha}\, \sqrt{1-\rho_t^2}\, d\bar{W}_t \tag{1}$$

while $\bar{\Gamma}_L$ and $\bar{\gamma}$ are the mean-reverting and reverting rate parameters respectively. The value $r > 0$ is the interest rate and $Z^{(j)}_t$ and $\bar{W}_t$ are Brownian background noises.
The payoff of a European exchange option with maturity at time $T > 0$ is:

$$h(S_T) = (S^{(1)}_T - S^{(2)}_T)_+ := max(S^{(1)}_T - S^{(2)}_T, 0)$$

On the other hand the price of an exchange contract with payoff above at time t, $0 \leq t \leq T$ and maturity at T is given by:

$$C_t = e^{-r(T-t)} E_{\mathcal{Q}}[h(S_T)] \tag{2}$$

Its terminal value is $C_T = h(S_T)$.
The price of the exchange contract at time t, $t < T$, depends on the behavior of $(V_s, \rho_s)_{t \leq s \leq T}$ summarized by the corresponding integrated processes on the interval $[t, T]$. It also depends on the spot prices, volatilities and correlation during the time interval to maturity. For simplicity in the notations we explicitly drop this last dependence. For the same reason, we analyze only the case $t = 0$.
Hence:

$$\begin{aligned} C_0 &= e^{-rT} \int_{\mathbb{R}^2 \times \Omega} h(x') f_{S_T, V_T^+, \rho_T^+}(x', x'')\, dx \\ &= e^{-rT} \int_{\Omega} \left[\int_{\mathbb{R}^2} C_T(x') f_{S_T/V_T^+, \rho_T^+}(x'/x'') \right] f_{V_T^+, \rho_T^+}(x'')\, dx'' \\ &= \int_{\Omega} C_M(x'') f_{V_T^+, \rho_T^+}(x'')\, dx'' \end{aligned} \tag{3}$$

where $\Omega = \mathbb{R}_+^2 \times (-1, 1)$ and $(x', x'') \in \mathbb{R}^2 \times \Omega$.
The function $C_M(x'') = e^{-rT} \int_{\mathbb{R}^2} C_T(x') f_{S_T/V_T^+, \rho_T^+}(x'/x'')\, dx'$ is the classic Margrabe price conditionally on $x'' = (V_T^+, \rho_T^+)$. After conditioning it equals the Margrabe price as obtained in [10]. A closed-form for the latter is given by:

$$C_M(V_T^+, \rho_T^+) = e^{-rT} S_0^{(1)} N(d_1(v_T^+)) - e^{-rT} S_0^{(2)} N(d_2(v_T^+)) \tag{4}$$

with:

$$\begin{aligned} d_1(v_T^+) &= \frac{\log\left(\frac{S_0^{(1)}}{S_0^{(2)}}\right) + \frac{1}{2} v_T^+}{\sqrt{v_T^+}} \\ d_2(v_T^+) &= \frac{\log\left(\frac{S_0^{(1)}}{S_0^{(2)}}\right) - \frac{1}{2} v_T^+}{\sqrt{v_T^+}} = d_1(v_T^+) - \sqrt{v_T^+} \end{aligned}$$

where:

$$v_T^+ = V_T^{1,+} + V_T^{2,+} - 2\sqrt{V_T^{1,+} V_T^{2,+}}\, \rho_T^+$$

and $(V_t^+)_{t \geq 0} = (V_t^{1,+}, V_t^{2,+})_{t \geq 0}$. The function $N(.)$ represents the cumulative distribution function of a standard normal random variable.
Next, to approximate the price in equation (2) we consider a second order Taylor expansion of the conditional Margrabe price $C_M(x), x \in \Omega$ around the average

volatilities and correlation values. It leads to:

$$\begin{aligned}
\hat{C}_M(x) &= C_M(x_0) + \frac{\partial C_M(x_0)}{\partial x_1}(x_1 - x_{0,1}) + \frac{\partial C_M(x_0)}{\partial x_2}(x_2 - x_{0,2}) + \frac{\partial C_M(x_0)}{\partial x_3}(x_3 - x_{0,3}) \\
&+ \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_1^2}(x_1 - x_{0,1})^2 + \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_2^2}(x_2 - x_{0,2})^2 \\
&+ \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_3^2}(x_3 - x_{0,3})^2 + \frac{\partial^2 C_M(x_0)}{\partial x_1 x_2}(x_1 - x_{0,1})(x_2 - x_{0,2}) \\
&+ \frac{\partial^2 C_M(x_0)}{\partial x_1 x_3}(x_1 - x_{0,1})(x_3 - x_{0,3}) + \frac{\partial^2 C_M}{\partial x_2 x_3}(x_0)(x_2 - x_{0,2})(x_3 - x_{0,3})
\end{aligned} \tag{5}$$

where $x_0 := (x_{0,1}, x_{0,2}, x_{0,3}) = (E_{\mathcal{Q}}(V_T^{1,+}), E_{\mathcal{Q}}(V_T^{2,+}), E_{\mathcal{Q}}(\rho_T^+))$ Hereby we assume the existence of the joint probability density function of the triplet integrated process (S_t, V_T^+, ρ_T^+), denoted by f_{S_t, V_T^+, ρ_T^+}. Combining equations (3) and (5) we have that the price C_0 is approximated by:

$$\begin{aligned}
\hat{C}_0 &= C_M(x_0) + \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_1^2} Var_{\mathcal{Q}}(V_T^{1,+}) + \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_2^2} Var_{\mathcal{Q}}(V_T^{2,+}) \\
&+ \frac{1}{2}\frac{\partial^2 C_M(x_0)}{\partial x_3^2} Var_{\mathcal{Q}}(\rho_T^+) + \frac{\partial^2 C_M(x_0)}{\partial x_1 x_2} cov_{\mathcal{Q}}(V_T^{1,+}, V_T^{2,+})
\end{aligned} \tag{6}$$

Notice that the Margrabe price $C_M(x'') \in C^\infty(\Omega)$, i.e. it has derivatives of any order on the set Ω.
Equation (5) is obtained substituting equation (5) into (3), taking into account that:

$$\begin{aligned}
\int_{\mathbb{R}^3} (x_1 - x_{0,1}) f_{V_T^+, \rho_T^+}(x)\, dx &= \int_{\mathbb{R}} (x_1 - x_{0,1}) \left[\int_{\mathbb{R}^2} f_{V_T^+, \rho_T^+}(x)\, dx_2\, x_3 \right] x_1\, dx_1 \\
&= E_{\mathcal{Q}}(V_T^{1,+} - E_{\mathcal{Q}}(V_T^{1,+})) = 0 \\
\int_{\mathbb{R}^3} (x_2 - x_{0,2}) f_{V_T^+, \rho_T^+}(x)\, dx &= \int_{\mathbb{R}} (x_2 - x_{0,2}) \left[\int_{\mathbb{R}^2} f_{V_T^+, \rho_T^+}(x)\, dx_1\, x_3 \right] dx_2 \\
&= E_{\mathcal{Q}}(V_T^{2,+} - E_{\mathcal{Q}}(V_T^{2,+})) = 0 \\
\int_{\mathbb{R}^3} (x_3 - x_{0,3}) f_{V_T^+, \rho_T^+}(x)\, dx &= \int_{\mathbb{R}} (x_3 - x_{0,3}) \left[\int_{\mathbb{R}^2} f_{V_T^+, \rho_T^+}(x)\, dx_1\, x_2 \right] dx_3 \\
&= E_{\mathcal{Q}}(\rho_T^+ - E_{\mathcal{Q}}(\rho_T^+)) = 0
\end{aligned}$$

The last result after assuming independence between volatility and correlation background noises.

Remark 2.1. *Sensitivities with respect to the parameters in the contract can be computed in a similar way. For example, an approximation of the deltas in the exchange contract are obtaining by differentiating equation (6) with respect to the initial price of the underlying assets.*

Computing derivatives of the Margrabe price with respect to the volatilities and correlation is straightforward. This issue is addressed in details in appendix B.
In order to estimate the option pricing function above we need to compute the

moments of $(V_T^{1,+}, V_T^{2,+}, \rho_T^+)$. To this end we introduce the following notations:

$$\begin{aligned} mr_j(t) &= E_{\mathcal{Q}}[\rho_t^j],\ mr_j^+(t) = E_{\mathcal{Q}}[(\rho_t^+)^j],\ j = 1,2 \\ mv_{j,k}(t) &= E_{\mathcal{Q}}[(V_t^{(k)})^j],\ mv_{j,k}^+(t) = E_{\mathcal{Q}}[(V_t^{k,+})^j]\ j,k = 1,2 \\ mv_{12}(t) &= E_{\mathcal{Q}}[V_t^{(1)} V_t^{(2)}],\ mv_{12}^+(t) = E_{\mathcal{Q}}[V_t^{1,+} V_t^{2,+}] \end{aligned}$$

Specific results are given below under a dynamic of an Ornstein-Ulenbeck process for volatilities. The latter are modeled as the Ornstein-Ulenbeck processes:

$$d\sigma_t^{(j)} = -\alpha_j \sigma_t^{(j)} dt + \beta_j dW_t^{(j)},\ j = 1,2 \tag{7}$$

where the Brownian motions $(W_t^{(1)})_{t\geq 0}$ and $(W_t^{(2)})_{t\geq 0}$ have instantaneous correlation ρ_V.
By Ito formula:

$$dV_t^{(j)} = c_j(V_L^{(j)} - V_t^{(j)})\, dt + \xi_j \sigma_t^{(j)}\, dW_t^{(j)},\ j = 1,2 \tag{8}$$

The parameters $V_L = (V_L^{(1)}, V_L^{(2)})$ and $c_j > 0$ are respectively the mean-reverting level and rate of the squared volatility processes.
The two components of the Brownian motion $(Z_t)_{t\geq 0} = (Z_t^{(1)},\ Z_t^{(2)})_{t\geq 0}$ are assumed to be independent of the second set of Brownian motions $(W_t)_{t\geq 0} = (W_t^{(1)},\ W_t^{(2)})_{t\geq 0}$ and $\bar{W}_t$. The Brownian motions of volatilities and correlation are also assumed to be independent.

Remark 2.2. *Parameters in models the (7) and (8) are related by* $c_j = 2\alpha_j$, $V_L^{(j)} = \frac{\beta_j^2}{2\alpha_j}$ *and* $\xi_j = 2\beta_j$.

Results regarding the moments on the integrated processes are given in the propositions below, while proofs are deferred to appendix A.

Proposition 2.3. *Let the correlation process* $(\rho_t)_{t\geq 0}$ *satisfy equation (1). Then:*

$$mr_1^+(t) := E_{\mathcal{Q}}(\rho_t^+) = \bar{\Gamma}_L t + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)(1 - e^{-\bar{\gamma} t}) \tag{9}$$

$$mr_2^+(t) := E_{\mathcal{Q}}[(\rho_t^+)^2] = b_0 + b_1 t + b_2 t^2 + b_3 t e^{-\bar{\gamma} t} + b_4 e^{-(2\bar{\gamma} + \bar{\alpha}^2)t} - (b_0 + b_4) c e^{-\bar{\gamma} t} \tag{10}$$

$$Var_{\mathcal{Q}}(\rho_t^+) = mr_2^+(t) - (mr_1^+(t))^2 \tag{11}$$

where:

$$\begin{aligned}
a_1 &= \frac{2\bar{\gamma}\bar{\Gamma}_L^2 + \bar{\alpha}^2}{2\bar{\gamma} + \bar{\alpha}^2}, a_2 = \frac{2\bar{\gamma}\bar{\Gamma}_L(\rho_0 - \bar{\Gamma}_L)}{\bar{\gamma} + \bar{\alpha}^2} \\
b_0 &= \frac{1}{\bar{\gamma}^2}\left(-a_1 + \rho_0^2 + 2\bar{\Gamma}_L(2-\rho) - \frac{a_2\alpha^2}{\bar{\gamma}} - \frac{\alpha^2(\rho_0^2 - a_1 - a_2)}{(2\bar{\gamma} + \bar{\alpha}^2)}\right) \\
b_1 &= \frac{1}{\bar{\gamma}}\left(2\bar{\Gamma}_L(\rho - 1) + \frac{\bar{\alpha}^2(1-a_1)}{\bar{\gamma}}\right) \\
b_2 &= \Gamma_L^2,\ b_3 = \frac{a_2}{\bar{\gamma}}\left(\frac{\bar{\alpha}^2}{\bar{\gamma}} - 1\right) \\
b_4 &= \frac{\rho_0^2 - a_1 - a_2}{\bar{\gamma}(\bar{\gamma} + \bar{\alpha}^2)}\left(1 - \frac{\bar{\alpha}^2}{2\bar{\gamma} + \bar{\alpha}^2}\right)
\end{aligned}$$

Second order moments and covariance of the integrated squared volatility are given in the propositions below. Previously we introduce the following constants:

$$\begin{aligned}
d_{0,j} &= (2c_jV_L^{(j)} + \xi_j^2)\frac{V_L^{(j)}}{2c_j} \\
d_{1,j} &= (2c_j + \xi_j^2)\frac{(V_0^{(j)} - V_L^{(j)})}{c_j} \\
g_{0,j} &= \frac{1}{c_j^2}\left[(V_0^{(j)} - V_L^{(j)})^2 + (V_L^{(j)})^2 - \frac{\xi_j^2 V_L^{(j)}}{c_j} - d_0\right] \\
g_{1,j} &= \frac{2}{c_j}\left(V_L^{(j)}(V_0^{(j)} - V_L^{(j)})\right) + \frac{\xi_j^2 V_L^{(j)}}{c_j^2} \\
g_{2,j} &= (V_L^{(j)})^2,\ g_{3,j} = -\frac{1}{c_j^2}\left[d_{0,j} + d_{1,j} - (V_0^{(j)})^2\right] \\
g_{4,j} &= -\frac{\xi_j^2}{c_j^2}\left(V_0^{(j)} - V_L^{(j)}\right)
\end{aligned}$$

Hence:

Proposition 2.4. *Let the process $(V_t)_{t\geq 0}$ satisfy equations (8). Then:*

$$mv_{1,j}^+(t) = V_L^{(j)}t + \frac{V_0^{(j)} - V_L^{(j)}}{c_j}(1 - e^{-c_jt}),\ j = 1,2 \tag{12}$$

$$mv_{2,j}^+(t) = P_{1,j}(t) - (g_{0,j} + g_{3,j} + \frac{d_1}{c_j})e^{-c_jt} + g_{3,j}e^{-2c_jt} + g_{4,j}te^{-c_jt},\ j = 1,2 \tag{13}$$

$$Var_{\mathcal{Q}}[V_t^{+,j}] = mv_{2,j}^+(t) - [mv_{1,j}^+(t)]^2,\ j = 1,2$$

with:

$$P_{1,j}(t) = g_{0,j} + g_{1,j}t + g_{2,j}t^2$$

Proposition 2.5. *Let the process $(V_t)_{t\geq 0}$ satisfy equation (8). Then:*

$$cov(V_t^{+,1}, V_t^{+,2}) = mv_{12}^+ - mv_{1,1}^+(t)mv_{1,2}^+(t)$$

where:

$$
\begin{aligned}
mv_{12}^{+}(t) &= E_{\mathcal{Q}}[V_t^{+,1}V_t^{+,2}] = \frac{1}{c_1c_2}\Big[P_2(t) - (V_0^{(1)} + c_1V_L^{(1)}t)mv_{1,2}(t) - (V_0^{(2)} + c_2V_L^{(2)}t)mv_{1,1}(t) \\
&+ ms_{12}(t) - \xi_1\xi_2\rho_V(e^{-c_1t}B_1(t) + e^{-c_2t}B_2(t)) + \xi_1\xi_2\rho_V A(t)\Big]
\end{aligned}
$$

where:

$$
\begin{aligned}
P_2(t) &= V_0^{(1)}V_0^{(2)} + (c_2V_0^{(1)}V_L^{(2)} + c_1V_0^{(2)}V_L^{(1)})t + c_1c_2V_L^{(1)}V_L^{(2)}t^2 \\
A(t) &= \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(t - \frac{2}{c_1+c_2}(1-e^{-\frac{1}{2}(c_1+c_2)t})\right) + \frac{2\sigma_0^{(1)}\sigma_0^{(2)}}{c_1+c_2}\left(1-e^{-\frac{1}{2}(c_1+c_2)t}\right) \\
B_j(t) &= \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left[\frac{1}{c_j}(e^{c_jt}-1) - \frac{2(-1)^j}{c_2-c_1}(e^{\frac{1}{2}(-1)^j(c_2-c_1)t}-1)\right] \\
&+ \sigma_0^{(1)}\sigma_0^{(2)}\frac{2(-1)^j}{c_2-c_1}\left(e^{\frac{1}{2}(-1)^j(c_2-c_1)t}-1\right),\ c_1 \neq c_2 \\
B_j(t) &= \frac{\xi_1\xi_2\rho_V}{4c_j}\left(\frac{1}{c_j}(e^{c_jt}-1)-1\right) + \sigma_0^{(1)}\sigma_0^{(2)}t,\ c_1 = c_2
\end{aligned}
$$

The functions $m_{1,j}^{+}(t)$ are given by equation (12) while:

$$
\begin{aligned}
ms_{12}(t) &= \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(1-e^{-\frac{1}{2}(c_1+c_2)t}\right) + \sigma_0^{(1)}\sigma_0^{(2)}e^{-\frac{1}{2}(c_1+c_2)t} \\
m_{1,j}(t) &= V_L^{(j)} + (V_0^{(j)} - V_L^{(j)})e^{-c_jt}
\end{aligned}
$$

3. Numerical pricing results

We consider an exchange contract between futures of West Intermediate Texas(WTI) and Brent oil types traded at NYSE. Both commodities exhibit similar patterns of behavior. Firstly, as is expected, they are highly correlated. The overall linear correlation of the series of daily future prices during the period of Dec 2013 and Jan 2019 is equal to 98%, while the correlation of the daily log-returns is 3.81%. On the other hand, when the correlation is rolled over sliding windows of 50 days it exhibits notable random variations with an apparent mean-reverting factor. This empirical fact suggests the presence of a random time-dependent correlation instead of a constant one, as classic models assume. See figures 1(a) and (b) for the linear correlation on prices and log-returns respectively.

A summary of the first four moments of the log-return series is shown in table 1. A high kurtosis indicates the presence of heavy-tailed distribution in both commodities.

Asset	Mean	Standard deviation	Skewness	Kurtosis
WTI	-0.0003	0.0211	0.1089	6.0696
Brent	-0.0004	0.0201	0.1473	5.9818

Table 1. First four moments of log-returns WTI and Brent

To illustrate the behavior of the elements in the model we consider a set of parameters as shown in table 2. As initial prices for both assets the values $S_0^{(1)} = 100, S_0^{(2)} = 96$ in US dollars are considered. Initial squared volatilities are $V_0 =$

(a) (b)

FIGURE 1. Left: Fifty days moving window correlation coefficient between WTI and Brent daily future prices . Right: Same window for the log-returns

(a) (b)

FIGURE 2. Simulated one-year length series of prices of both commodities(left) generated by a Monte Carlo method from the model with stochastic correlation and the corresponding realizations of their squared volatilities(right)

$(0.1, 0.1)$ or about 31,6 % of volatility in both assets. Notice from table 1 that both WTI and Brent commodities present similar standard deviations. The initial correlation is $\rho_0 = -0.3$. The parameters in the model for the squared volatilities in equation (8) are related, see remark (2.2). As a consequence we have that $V_L^{(j)} = \frac{\xi_j^2}{4c_j}$.

The correlation between the Brownian motions driving the volatilities dynamic is $\rho_v = 0.80$, reflecting a high joint fluctuation as suggested by the data. The mean-reverting levels and rates of the volatility processes are $V_L = (1/8, 1/8)$ and $c = (1, 1)$ respectively. The later indicates 6 months of reverting rate. Analogous parameters in the correlation processes are $\bar{\Gamma} = 0.8$ and $\bar{\gamma} = 0.8$. The annual interest rate is $r = 4\%$, and the maturity time is one year.

Parameters have been chosen for illustrative proposes in a way that the simulated oil prices fall within the range of those observed in the data. For a discussion about parameter estimation based on a *Generalized Method of Moments* on a related model having for underlying commodities oil and gasoline within the context of spread pricing see [14].

Asset Component	WTI sqr. vol.	Brent sqr. vol.	Correlation
MR level	$V_L^{(1)} = \frac{1}{8}$	$V_L^{(2)} = \frac{1}{8}$	$\bar{\Gamma}_L = 0.8$
MR rate	$c_1 = 1$	$c_2 = 1$	$\bar{\gamma} = 0.8$
vol.	$\xi_1 = 0.5$	$\xi_2 = 0.5$	$\bar{\alpha} = 1$
Initial values	$V_0^{(1)} = 0.1$	$V_0^{(2)} = 0.1$	$\rho_0 = -0.3, \rho_V = 0.8$

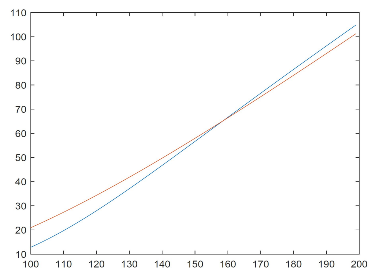
TABLE 2. Parametric set for the squared volatilities and correlation models

The trajectories of the two processes from a Monte Carlo simulation are shown in figure 2. Figure 2(a) represents the series of prices while figure 2(b) shows a realization of the squared volatilities. Simulation has been coded in MATLAB computer language with an Intel(R) Core(TM) i7-10510U CPU after implementing an Euler-Maruyama scheme on the corresponding stochastic differential equations. A second Taylor approach took an average of 2.9078×10^{-4} seconds while an alternative Monte Carlo method with 10^4 repetitions necessitated 118.050411 seconds.

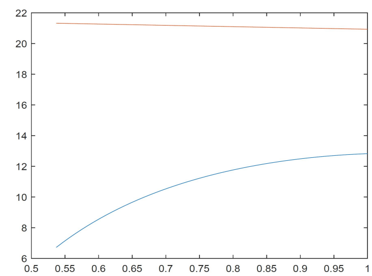
Sensitivities, popularly known as *greeks*, in the exchange contract value with respect to the initial prices of the underlying commodities, given in figure 3(a), the time to maturity, in figure 3(b) and the initial correlation between underlying assets in figure 3(c) are shown. All the parameters in the model and contract in table 2 have been kept constant except the one considered in this empirical sensitivity analysis. In the three figures the blue curve represents the Margrabe price under the model with stochastic correlation and volatilities while the red curve signals the Margrabe price in the classic setting. All prices have been computed using a second order Taylor expansion. Thus, in figure 3(a) the initial price of one of the underlying commodities is moved within an interval of 100-200 dollars while the other is kept constant at 96\$. The value of the exchange increases under both models with the increments in the difference between the starting underlying prices, but in the model with stochastic correlation the rate of the increments is higher. Notice also that when the contract is *on-the-money* the value of the classic exchange contract outperforms the one with stochastic correlation, while when the contract is *out-the-money* the situation reverses. It illustrates a possible overvaluation or undervaluation of exchange contracts as a result of neglecting random variations in the correlation and volatilities.

In figure 3(b) the price of the contract when maturity time ranges between six months and one year is shown. It is observed that the price under the stochastic correlation model stabilizes as maturity increases indicating a possible Law of Large Number or ergodic effect, i.e. random variations in the correlation are compensated in the long run. In the classic Margrabe model the price slowly decreases with maturity time.

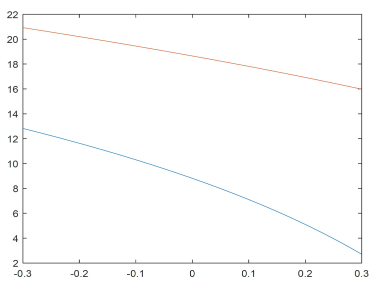
In figure 3(c) the price of the exchange is shown as function of different initial correlations. As expected, the price decreases with higher volatilities. Moreover, a negative correlation tends to favor the separation of the underlying commodity



(a)



(b)



(c)

FIGURE 3. Figure (a) shows the influence in the prices of an exchange contract with respect to initial prices. Figure (b) exhibits the behavior of the price as function of the time to maturity with figure (c) reflects the sensitivities regarding the initial correlation between underlying assets

prices at maturity under both models. Again, the classic Margrabe model overestimates the price.
On the other hand, figure 4 shows the price as function of the squared volatilities of both assets in the case of the model with stochastic correlations. The price increases as volatilities increases as expected. Figure 5(a) shows a comparison between the conditional Margrabe price as function of the squared volatility of the first commodity(red line) for a selected interval around the average squared volatility and the first(blue line) and second Taylor(dotted yellow line) approximations. As it can be observed a second order Taylor approximation offers a suitable approximation of the Margrabe price.
In figure 5(b) the difference between Margrabe price and a second Taylor approximation for a set of squared volatilities in the first and second commodities is shown. Again, the numerical results show a reasonable approximation except at volatilities near the origin, which are not expected in most realistic situations.

4. CONCLUSIONS

A second Taylor approximation offers a suitable method to price exchange contracts beyond the Margrabe's classic framework when stochastic volatilities and

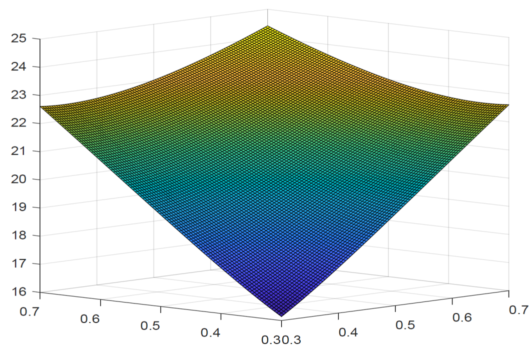


FIGURE 4. A change in the prices of an exchange contract with respect to squared volatilities

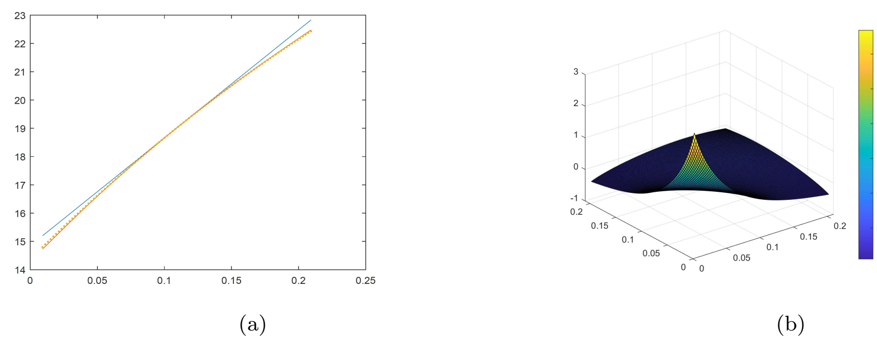


FIGURE 5. Conditional Margrabe price vs Taylor approximations of first and second order in figure (a). The difference between conditional Margrabe price as function of both volatilities is shown in figure (b).

correlation between them are taken into account. In the parametric set considered it produces accurate results with significant less computational effort than a standard Monte Carlo approach. In comes with the caveat that the Taylor approximation is fairly accurate around the mean values of volatilities and correlation, which in practical situations need to be priory assessed.
The sensitivity analysis produces an interesting insight about risk factors affecting the price of the exchange, while the comparison with the classic Margrabe setting shows a possible overestimation of the price of the exchange when stochastic correlation and volatilities are not taken into account.

5. Acknowledgement

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6. Appendix

6.1. Appendix A: Moments of the volatility and correlation. Proof of proposition 2.3

Proof. For the first moment notice that:

$$\rho_t = \rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t - \bar{\gamma}\int_0^t \rho_s\, ds + \bar{\alpha}\int_0^t \sqrt{1-\rho_s^2}\, d\bar{W}_s \tag{14}$$

Taking expected value on both sides:

$$mr_1(t) := E_{\mathcal{Q}}(\rho_t) = \rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t - \bar{\gamma}\int_0^t mr_1(s)\, ds$$

Differentiating we get:

$$mr_1'(t) = \bar{\gamma}\bar{\Gamma}_L - \bar{\gamma} mr_1(t)$$

whose solution is:

$$mr_1(t) = \bar{\Gamma}_L + (\rho_0 - \bar{\Gamma}_L)e^{-\bar{\gamma}t}$$

Hence, for the integrated process:

$$\begin{aligned}
E_{\mathcal{Q}}(\rho_t^+) &= \int_0^t \bar{\Gamma}_L + (\rho_0 - \bar{\Gamma}_L)e^{-\bar{\gamma}s}\, ds \\
&= \bar{\Gamma}_L t + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)(1 - e^{-\bar{\gamma}t})
\end{aligned}$$

To compute the second moment we first apply Ito formula to $f(x) = x^2$ and the correlation process. Hence:

$$\begin{aligned}
\rho_t^2 &= \rho_0^2 + 2\int_0^t \rho_s d\rho_s + < \rho_t > \\
&= \rho_0^2 + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t \rho_s\, ds - 2\bar{\gamma}\int_0^t \rho_s^2\, ds + 2\bar{\alpha}\int_0^t \rho_s\sqrt{1-\rho_s^2}\, d\bar{W}_s \\
&+ \bar{\alpha}^2 \int_0^t (1-\rho_s^2)\, ds \\
&= \rho_0^2 + \bar{\alpha}^2 t + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t \rho_s\, ds - (\bar{\alpha}^2 + 2\bar{\gamma})\int_0^t \rho_s^2\, ds \\
&+ 2\bar{\alpha}\int_0^t \rho_s\sqrt{1-\rho_s^2}\, d\bar{W}_s
\end{aligned}$$

Taking expected value:

$$E_{\mathcal{Q}}(\rho_t^2) = \rho_0^2 + \bar{\alpha}^2 t + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t E_{\mathcal{Q}}(\rho_s)\, ds - (2\bar{\gamma} + \bar{\alpha}^2)\int_0^t E_{\mathcal{Q}}(\rho_s^2)\, ds$$

After differentiating:

$$\begin{aligned} mr_2'(t) &+ (2\bar{\gamma}+\bar{\alpha}^2)mr_2(t) = 2\bar{\gamma}\bar{\Gamma}_L mr_1(t)+\bar{\alpha}^2 \\ mr_2(0) &= \rho_0^2 \end{aligned}$$

The solution of the equation above is:

$$mr_2(t) = a_1 + a_2 e^{-\bar{\gamma}t} + (\rho_0^2 - a_1 - a_2)e^{-(2\bar{\gamma}+\bar{\alpha}^2)t} \tag{15}$$

where the constants a_1 and a_2 have been defined in proposition 2.3.
Next, notice that we have:

$$\frac{dmr_2^+}{dt} = 2E_{\mathcal{Q}}[\rho_t^+\rho_t]$$

From equation (14):

$$E_{\mathcal{Q}}(\rho_t + \bar{\gamma}\rho_t^+)^2 = E_{\mathcal{Q}}(\rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t + \bar{\alpha}\int_0^t \sqrt{1-\rho_s^2}\,d\bar{W}_s)^2$$

Expanding both sides in the equation above we have:

$$\begin{aligned} LHS = E_{\mathcal{Q}}(\rho_t + \bar{\gamma}\rho_t^+)^2 &= E_{\mathcal{Q}}(\rho_t^2) + 2\bar{\gamma}E_{\mathcal{Q}}(\rho_t\rho_t^+) + \bar{\gamma}^2 E_{\mathcal{Q}}(\rho_t^+)^2 \\ &= mr_2(t) + \bar{\gamma}\frac{dmr_2^+}{dt} + \bar{\gamma}^2 mr_2^+(t) \end{aligned}$$

and

$$\begin{aligned} RHS &= (\rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t)^2 + 2(\rho_0 + 2\bar{\gamma}\,\bar{\Gamma}_L t)\bar{\alpha}E_{\mathcal{Q}}(\int_0^t \sqrt{1-\rho_s^2}\,d\bar{W}_s) \\ &+ \bar{\alpha}^2 E_{\mathcal{Q}}\left(\int_0^t \sqrt{1-\rho_s^2}\,d\bar{W}_s\right)^2 \\ &= (\rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t)^2 + \bar{\alpha}^2 E_{\mathcal{Q}}(\int_0^t \sqrt{1-\rho_s^2}\,d\bar{W}_s)^2 \\ &= (\rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t)^2 + \bar{\alpha}^2 E_{\mathcal{Q}}(\int_0^t (1-\rho_s^2)\,ds) \\ &= (\rho_0 + \bar{\gamma}\,\bar{\Gamma}_L t)^2 + \bar{\alpha}^2(t - \int_0^t mr_2(s)\,ds) \end{aligned}$$

From equation (15):

$$\begin{aligned} \int_0^t mr_2(s)\,ds &= \int_0^t (a_1 + a_2 e^{-\bar{\gamma}s} + (\rho_0^2 - a_1 - a_2)e^{-(2\bar{\gamma}+\bar{\alpha}^2)s})\,ds \\ &= a_1 t + \frac{a_2}{\bar{\gamma}}(1 - e^{-\bar{\gamma}t}) + \frac{\rho_0^2 - a_1 - a_2}{2\bar{\gamma}+\bar{\alpha}^2}(1 - e^{-(2\bar{\gamma}+\bar{\alpha}^2)t}) \end{aligned}$$

Hence,

$$\frac{dmr_2^+}{dt} + \bar{\gamma}mr_2^+(t) = b(t)$$

where:

$$\begin{aligned} b(t) &= \frac{1}{\bar\gamma}\left(-mr_2(t)+(\rho_0+\bar\gamma\,\bar\Gamma_L t)^2+\bar\alpha^2 t-\frac{\bar\alpha^2}{\bar\gamma}\int_0^t mr_2(s)\,ds\right)\\ &= \frac{1}{\bar\gamma}\left(-mr_2(t)+(\rho_0+\bar\gamma\,\bar\Gamma_L t)^2+\bar\alpha^2 t\right)\\ &- \frac{1}{\bar\gamma}\left(a_1 t+\frac{a_2}{\bar\gamma}(1-e^{-\bar\gamma t})+\frac{\rho_0^2-a_1-a_2}{2\bar\gamma+\bar\alpha^2}(1-e^{-(2\bar\gamma+\bar\alpha^2)t})\right) \end{aligned}$$

and initial condition $mr_2^+(0)=0$.
Using the integrating factor $e^{\bar\gamma t}$ we find that its solution is:

$$mr_2^+(t) = e^{-\bar\gamma t}\int e^{\bar\gamma t} b(t)\,dt + ce^{-\bar\gamma t} \tag{16}$$

But:

$$\begin{aligned} \int e^{\bar\gamma t} b(t)\,dt &= \frac{1}{\bar\gamma}\int e^{\bar\gamma t}\left(-mr_2(t)+(\rho_0+\bar\gamma\,\bar\Gamma_L t)^2+\bar\alpha^2 t\right)\,dt\\ &- \frac{1}{\bar\gamma}\int e^{\bar\gamma t}\left(a_1 t+\frac{a_2}{\bar\gamma}(1-e^{-\bar\gamma t})+\frac{\rho_0^2-a_1-a_2}{2\bar\gamma+\bar\alpha^2}(1-e^{-(2\bar\gamma+\bar\alpha^2)t})\right)\,dt \end{aligned}$$

Moreover, from equation (15):

$$\begin{aligned} \int e^{\bar\gamma t} mr_2(t)\,dt &= \int e^{\bar\gamma t}(a_1+a_2 e^{-\bar\gamma t}+(\rho_0^2-a_1-a_2)e^{-(2\bar\gamma+\bar\alpha^2)t})\,dt\\ &= \frac{a_1}{\bar\gamma}e^{\bar\gamma t}+a_2 t-\frac{\rho_0^2-a_1-a_2}{\bar\gamma+\bar\alpha^2}e^{-(\bar\gamma+\bar\alpha^2)t}\\ \int(\rho_0+\bar\gamma\,\bar\Gamma_L t)^2 e^{\bar\gamma t}\,dt &= \frac{\rho_0^2}{\bar\gamma}e^{\bar\gamma t}+2\bar\gamma\bar\Gamma_L\rho_0(\frac{1}{\bar\gamma}te^{\bar\gamma t}-\frac{1}{\bar\gamma^2}e^{\bar\gamma t})\\ &+ \bar\gamma^2\bar\Gamma_L^2(\frac{1}{\bar\gamma}t^2e^{\bar\gamma t}-\frac{2}{\bar\gamma^2}te^{\bar\gamma t}+\frac{2}{\bar\gamma^3}e^{\bar\gamma t})\\ &= \left[\rho_0^2+2\bar\gamma\bar\Gamma_L\rho_0(t-\frac{1}{\bar\gamma})+\bar\gamma^2\bar\Gamma_L^2(t^2-\frac{2}{\bar\gamma}t+\frac{2}{\bar\gamma^2})\right]\frac{1}{\bar\gamma}e^{\bar\gamma t}\\ \int te^{\bar\gamma t}\,dt &= \frac{1}{\bar\gamma}\left(t-\frac{1}{\bar\gamma}\right)e^{\bar\gamma t} \end{aligned}$$

Hence:

$$\begin{aligned} \int e^{\bar\gamma t} b(t)\,dt &= -\frac{1}{\bar\gamma^2}a_1 e^{\bar\gamma t}-\frac{a_2 t}{\bar\gamma}+\left(\frac{\rho_0^2-a_1-a_2}{\bar\gamma(\bar\gamma+\bar\alpha^2)}\right)e^{-(\bar\gamma+\bar\alpha^2)t}\\ &+ \frac{1}{\bar\gamma^2}\left[\rho_0^2+2\bar\gamma\bar\Gamma_L\rho_0(t-\frac{1}{\bar\gamma})+\bar\gamma^2\bar\Gamma_L^2(t^2-\frac{2}{\bar\gamma}t+\frac{2}{\bar\gamma^2})\right]e^{\bar\gamma t}\\ &+ \frac{\bar\alpha^2}{\bar\gamma^2}\left(t-\frac{1}{\bar\gamma}\right)e^{\bar\gamma t}\\ &- \frac{\bar\alpha^2}{\bar\gamma}\left[\frac{a_1}{\bar\gamma}(t-\frac{1}{\bar\gamma^2})e^{\bar\gamma t}+\frac{a_2}{\bar\gamma^2}e^{\bar\gamma t}-\frac{a_2}{\bar\gamma}t\right.\\ &+ \left.\frac{\rho_0^2-a_1-a_2}{\bar\gamma(2\bar\gamma+\bar\alpha^2)}e^{\bar\gamma t}+\frac{\rho_0^2-a_1-a_2}{(2\bar\gamma+\bar\alpha^2)(\bar\gamma+\bar\alpha^2)}e^{-(\bar\gamma+\bar\alpha^2)t}\right] \end{aligned}$$

Combining the expressions above into equation (16) we have (9-10).
On the other hand from the initial conditions $c=-b_0-b_4$.

From the first and second moments of ρ_t^+ we obtain the expression for the variance. □

Proof of proposition 2.4

Proof. To compute the first and second moments we proceed similarly to the proof of proposition 2.3. Notice equations for squared volatilities are of mean-reverting square root type s.d.e's as well.
Hence:

$$\begin{aligned} mv_{1,j}(t) &= V_L^{(j)} + (V_0^{(j)} - V_L^{(j)})e^{-c_j t} \\ mv_{1,j}^+(t) &= E_{\mathcal{Q}}[V_t^{+,j}] = V_L^{(j)} t + \frac{V_0^{(j)} - V_L^{(j)}}{c_j}(1 - e^{-c_j t}) \end{aligned}$$

Moreover,

$$\begin{aligned} (V_t^{(j)})^2 &= (V_0^{(j)})^2 + 2\int_0^t V_s^{(j)} dV_s^{(j)} + < V_t^{(j)} > \\ &= (V_0^{(j)})^2 + 2c_j V_L^{(j)} V_t^{j,+} - 2c_j \int_0^t (V_s^{(j)})^2 \, ds + 2\xi_j \int_0^t V_s^{(j)} \sigma_s^{(j)} \, dW_s^{(j)} \\ &+ \xi_j^2 V_t^{j,+} \\ &= (V_0^{(j)})^2 + (2c_j V_L^{(j)} + \xi_j^2) V_t^{j,+} - 2c_j \int_0^t (V_s^{(j)})^2 \, ds \\ &+ 2\xi_j \int_0^t V_s^{(j)} \sigma_s^{(j)} \, dW_s^{(j)} \end{aligned}$$

Taking expected value on both sides:

$$mv_{2,j}(t) = (V_0^{(j)})^2 + (2c_j V_L^{(j)} + \xi_j^2) \int_0^t mv_{1,j}(s) \, ds - 2c_j \int_0^t mv_{2,j}(s) \, ds$$

or

$$\begin{aligned} mv'_{2,j}(t) &+ 2c_j mv_{2,j}(t) = (2c_j V_L^{(j)} + \xi_j^2) mv_{1,j}(t) \\ mv_{2,j}(0) &= (V_0^{(j)})^2 \end{aligned}$$

Denoting by $c(t) = (2c_j V_L^{(j)} + \xi_j^2) mv_{1,j}(t)$ the solution of the ODE above is:

$$mv_{2,j}(t) = e^{-2c_j t} \int e^{2c_j t} c(t) \, dt + d_2 e^{-2c_j t}$$

But:

$$\begin{aligned} \int e^{2c_j t} c(t) \, dt &= (2c_j V_L^{(j)} + \xi_j^2) \int e^{2c_j t} mv_{1,j}(t) \, dt \\ &= (2c_j V_L^{(j)} + \xi_j^2) \int e^{2c_j t} (V_L^{(j)} + (V_0^{(j)} - V_L^{(j)}) e^{-c_j t}) \, dt \\ &= (2c_j V_L^{(j)} + \xi_j^2) \left(\frac{V_L^{(j)}}{2c_j} e^{2c_j t} + \frac{V_0^{(j)} - V_L^{(j)}}{c_j} e^{c_j t} \right) \end{aligned}$$

Then:

$$mv_{2,j}(t) = d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}$$

where:

$$d_2 = (V_0^{(j)})^2 - d_0 - d_1$$

Next, notice that we have:

$$\frac{dmv_{2,j}^{+}}{dt} = 2E_{\mathcal{Q}}[V_t^{(j,+)}V_t^{(j)}] \tag{17}$$

Now:

$$\begin{aligned} E_{\mathcal{Q}}(V_t^{(j)} + c_j V_t^{j,+})^2 &= E_{\mathcal{Q}}[V_0^{(j)} + c_j\, V_L^{(j)} t + \xi_j \int_0^t \sigma_s^{(j)}\, dW_s^{(j)}]^2 \\ &= (V_0^{(j)} + c_j\, V_L^{(j)} t)^2 + 2(V_0^{(j)} + c_j V_L^{(j)} t)\xi_j E_{\mathcal{Q}}\left(\int_0^t \sigma_s^{(j)}\, dW_s^{(j)}\right) \\ &+ \xi_j^2 E_{\mathcal{Q}}\left(\int_0^t \sigma_s^{(j)}\, dW_s^{(j)}\right)^2 \\ &= (V_0^{(j)} + c_j\, V_L^{(j)} t)^2 + \xi_j^2 \int_0^t mv_{1,j}(s)\, ds \qquad (18) \end{aligned}$$

On the other hand, after taking into account equation (17):

$$E_{\mathcal{Q}}(V_t^{(j)} + c_j V_t^{j,+})^2 = mv_{2,j}(t) + c_j \frac{dmv_{2,j}^{+}}{dt} + c_j^2 mv_{2,j}^{+} \tag{19}$$

Hence, equating equations (18) and (19) we have that $mv_{2,j}^{+}$ satisfies:

$$\begin{aligned} \frac{dmv_{2,j}^{+}}{dt} + c_j mv_{2,j}^{+}(t) &= d(t) \\ mv_{2,j}^{+}(0) &= 0 \end{aligned} \tag{20}$$

with:

$$\begin{aligned} d(t) &= \frac{(V_0^{(j)} + c_j\, V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} \int_0^t mv_{1,j}(s)\, ds - \frac{1}{c_j} mv_{2,j}(t) \\ &= \frac{(V_0^{(j)} + c_j\, V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} \int_0^t [V_L^{(j)} + (V_0^{(j)} - V_L^{(j)}) e^{-c_j t}]\, ds \\ &- \frac{1}{c_j}(d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \\ &= \frac{(V_0^{(j)} + c_j\, V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} V_L^{(j)} t - \frac{\xi_j^2}{c_j^2}(V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \\ &- \frac{1}{c_j}(d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \end{aligned}$$

The solution of equation (20) is:

$$\begin{aligned}
mv_{2,j}^{+}(t) &= e^{-c_j t}\int e^{c_j t} d(t)\, dt + ce^{-c_j t} \\
&= e^{-c_j t}\int e^{c_j t}\left[\frac{(V_0^{(j)} + c_j V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} V_L^{(j)} t - \frac{\xi_j^2}{c_j^2}(V_0^{(j)} - V_L^{(j)})e^{-c_j t}\right. \\
&- \left.\frac{1}{c_j}(d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t})\right]\, dt + ce^{-c_j t} \\
&= \frac{e^{-c_j t}}{c_j}\int e^{c_j t}\left[(V_0^{(j)} + c_j V_L^{(j)} t)^2 + \xi_j^2 V_L^{(j)} t - \frac{\xi_j^2}{c_j}(V_0^{(j)} - V_L^{(j)})e^{-c_j t}\right. \\
&- \left.(d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t})\right]\, dt + ce^{-c_j t}
\end{aligned}$$

(21)

Moreover:

$$\begin{aligned}
&\int e^{c_j t}(V_0^{(j)} + c_j V_L^{(j)} t)^2\, dt = \frac{(V_0^{(j)})^2}{c_j}e^{c_j t} + 2c_j V_L^{(j)}\int te^{c_j t}\, dt + c_j^2 (V_L^{(j)})^2\int t^2 e^{c_j t}\, dt \\
= &\frac{(V_0^{(j)})^2}{c_j}e^{c_j t} + 2c_j V_0^{(j)} V_L^{(j)}\left(\frac{t}{c_j}e^{c_j t} - \frac{1}{c_j^2}e^{c_j t}\right) + c_j^2 (V_L^{(j)})^2\left(\frac{t^2}{c_j}e^{c_j t} - \frac{2t}{c_j^2}e^{c_j t} + \frac{2}{c_j^3}e^{c_j t}\right) \\
= &\frac{e^{c_j t}}{c_j}\left[c_j^2 (V_L^{(j)})^2 t^2 + 2(c_j V_0^{(j)} V_L^{(j)} - c_j (V_L^{(j)})^2)t + (V_0^{(j)})^2 - 2V_0^{(j)} V_L^{(j)} + 2(V_L^{(j)})^2\right] \\
= &\frac{e^{c_j t}}{c_j}\left[c_j^2 (V_L^{(j)})^2 t^2 + 2c_j (V_0^{(j)} V_L^{(j)} - (V_L^{(j)})^2)t + (V_0^{(j)} - V_L^{(j)})^2 + (V_L^{(j)})^2\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int te^{c_j t}\xi_j^2 V_L^{(j)}\, dt = \frac{\xi_j^2 V_L^{(j)}}{c_j}(t - \frac{1}{c_j})e^{c_j t} \\
&\int e^{c_j t}\frac{\xi_j^2}{c_j}(V_0^{(j)} - V_L^{(j)})e^{-c_j t}\, dt = \frac{\xi_j^2}{c_j}(V_0^{(j)} - V_L^{(j)})t \\
&\int e^{c_j t}(d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t})\, dt = \frac{d_0}{c_j}e^{c_j t} + d_1 t - \frac{d_2}{c_j}e^{-c_j t} \\
= &\frac{e^{c_j t}}{c_j}[d_0 + d_1 c_j e^{-c_j t} - d_2 e^{-2c_j t}] \\
= &\frac{e^{c_j t}}{c_j}[d_0(1 + e^{-2c_j t}) + d_1 e^{-c_j t}(c_j + e^{-c_j t}) - (V_0^{(j)})^2 e^{-2c_j t}]
\end{aligned}$$

From the initial conditions:

$$c = -(g_{0,j} + g_{3,j} + g_{4,j})$$

Therefore, substituting in equation (21) we obtain (13). □

Proof of proposition 2.5

Proof. To compute the covariance of the integrated squared volatilities we start noticing that $<\sigma_t^{(1)}, \sigma_t^{(2)}>= \beta_1\beta_2\rho_V t$. Therefore, by Ito integration by parts formula:

$$\begin{aligned}
\sigma_t^{(1)}\sigma_t^{(2)} &= \sigma_0^{(1)}\sigma_0^{(2)} + \int_0^t \sigma_s^{(1)} d\sigma_s^{(2)} + \int_0^t \sigma_s^{(2)} d\sigma_t^{(1)} + <\sigma_t^{(1)}, \sigma_t^{(2)}> \\
&= \sigma_0^{(1)}\sigma_0^{(2)} - \alpha_2 \int_0^t \sigma_s^{(1)}\sigma_s^{(2)} ds + \beta_2 \int_0^t \sigma_s^{(1)} dW_t^{(2)} \\
&- \alpha_1 \int_0^t \sigma_s^{(1)}\sigma_s^{(2)} ds + \beta_1 \int_0^t \sigma_s^{(2)} dW_t^{(1)} + \beta_1\beta_2\rho_V t
\end{aligned}$$

Taking expected value on both sides:

$$E_{\mathcal{Q}}[\sigma_t^{(1)}\sigma_t^{(2)}] = \sigma_0^{(1)}\sigma_0^{(2)} - (\alpha_1+\alpha_2)\int_0^t E_{\mathcal{Q}}[\sigma_s^{(1)}\sigma_s^{(2)}]\, ds + \beta_1\beta_2\rho_V t$$

The expression above leads to the differential equation:

$$ms_{12}'(t) + (\alpha_1+\alpha_2) ms_{12}(t) - \beta_1\beta_2\rho_V = 0$$

with $ms_{12}(t) = E_{\mathcal{Q}}[\sigma_t^{(1)}\sigma_t^{(2)}]$ and $ms_{12}(0) = \sigma_0^{(1)}\sigma_0^{(2)}$.
Its solution is:

$$ms_{12}(t) = \frac{\beta_1\beta_2\rho_V}{\alpha_1+\alpha_2}\left(1 - e^{-(\alpha_1+\alpha_2)t}\right) + \sigma_0^{(1)}\sigma_0^{(2)} e^{-(\alpha_1+\alpha_2)t}$$

With the reparametrization in remark 2.2 it becomes:

$$ms_{12}(t) = \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(1 - e^{-\frac{1}{2}(c_1+c_2))}\right) + \sigma_0^{(1)}\sigma_0^{(2)} e^{-\frac{1}{2}(c_1+c_2)t} \tag{22}$$

Moreover, from equation (8) and Ito formula:

$$\begin{aligned}
V_t^{(1)}V_t^{(2)} &= V_0^{(1)}V_0^{(2)} + \int_0^t V_s^{(1)} dV_s^{(2)} + \int_0^t V_s^{(2)} dV_s^{(1)} + <V_t^{(1)}, V_t^{(2)}> \\
&= V_0^{(1)}V_0^{(2)} + c_2 V_L^{(2)} t - c_2 \int_0^t V_s^{(1)} V_s^{(2)} ds + \xi_2 \int_0^t V_s^{(1)}\sigma_s^{(2)} dW_t^{(2)} \\
&+ c_1 V_L^{(1)} t - c_1 \int_0^t V_s^{(2)} V_s^{(1)} ds + \xi_1 \int_0^t V_s^{(2)}\sigma_s^{(1)} dW_t^{(1)} + \xi_1\xi_2\rho_V \int_0^t \sigma_s^{(1)}\sigma_s^{(2)} ds
\end{aligned}$$

Again, taking expected value on both sides of the equation above and differentiating:

$$mv_{12}'(t) = c_1 V_L^{(1)} + c_2 V_L^{(2)} - (c_1+c_2) mv_{12}(t) + \xi_1\xi_2\rho_V ms_{12}(t)$$

whose solution is given by:

$$
\begin{aligned}
mv_{12}(t) &= \xi_1\xi_2\rho_V e^{-(c_1+c_2)t}\int_0^t e^{(c_1+c_2)s} ms_{12}(s)\,ds + ce^{-(c_1+c_2)t} \\
&= \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1+c_2)} e^{-(c_1+c_2)t}\int_0^t e^{(c_1+c_2)s}\left(1-e^{-\frac{1}{2}(c_1+c_2)s}\right)\,ds \\
&+ \sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V e^{-(c_1+c_2)t}\int_0^t e^{(c_1+c_2)s}e^{-\frac{1}{2}(c_1+c_2)s}\,ds + ce^{-(c_1+c_2)t} \\
&= \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1+c_2)} e^{-(c_1+c_2)t}\int_0^t e^{(c_1+c_2)s}\,ds - \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1+c_2)} e^{-(c_1+c_2)t}\int_0^t e^{\frac{1}{2}(c_1+c_2)s}\,ds \\
&+ \sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V e^{-(c_1+c_2)t}\int_0^t e^{\frac{1}{2}(c_1+c_2)s}\,ds + ce^{-(c_1+c_2)t} \\
&= \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1+c_2)^2} - \frac{(\xi_1\xi_2\rho_V)^2}{(c_1+c_2)^2}e^{-\frac{1}{2}(c_1+c_2)t} + \frac{2\sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V}{c_1+c_2}e^{-\frac{1}{2}(c_1+c_2)t} + ce^{-(c_1+c_2)t}
\end{aligned}
$$

From the initial condition $mv_{12}(0) = V_0^{(1)}V_0^{(2)}$ we have that:

$$
c = V_0^{(1)}V_0^{(2)} + \frac{1}{2}\frac{(\xi_1\xi_2\rho_V)^2}{(c_1+c_2)^2} - \frac{2\sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V}{c_1+c_2}
$$

On the other hand, from equation (8):

$$
\begin{aligned}
V_t^{j,+} &= \frac{1}{c_j}[V_0^{(j)} + c_1V_L^{(j)}t - V_t^{(j)} + \xi_j\int_0^t \sigma_t^{(j)}\,dW_t^{(j)}] \\
V_t^{(j)} &= V_0^{(j)}e^{-c_jt} + V_L^{(j)}(1-e^{-c_jt}) + \xi_j e^{-c_jt}\int_0^t e^{c_js}\sigma_s^{(j)}\,dW_t^{(j)}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
mv_{12}^{+}(t) &:= E_{\mathcal{Q}}[V_t^{1,+}V_t^{2,+}] \\
&= \frac{1}{c_1c_2}E_{\mathcal{Q}}[(V_0^{(1)} + c_1V_L^{(1)}t - V_t^{(1)} + \xi_1\int_0^t \sigma_s^{(1)}\,dW_s^{(1)})(V_0^{(2)} + c_2V_L^{(2)}t - V_t^{(2)} + \xi_2\int_0^t \sigma_s^{(2)}\,dW_s^{(2)})] \\
&= \frac{1}{c_1c_2}\Big[V_0^{(1)}V_0^{(2)} + c_2V_0^{(1)}V_L^{(2)}t - V_0^{(1)}E_{\mathcal{Q}}[V_t^{(2)}] + \xi_2V_0^{(1)}E_{\mathcal{Q}}[\int_0^t \sigma_s^{(2)}\,dW_s^{(2)}] \\
&+ c_1V_0^{(2)}V_L^{(1)}t + c_1c_2V_L^{(1)}V_L^{(2)}t^2 - c_1V_L^{(1)}tE_{\mathcal{Q}}[V_t^{(2)}] + c_1\xi_2V_L^{(1)}tE_{\mathcal{Q}}[\int_0^t \sigma_s^{(2)}\,dW_s^{(2)}] \\
&- V_0^{(2)}E_{\mathcal{Q}}[V_t^{(1)}] - c_2V_L^{(2)}tE_{\mathcal{Q}}[V_t^{(1)}] + E_{\mathcal{Q}}[V_t^{(1)}V_t^{(2)}] - \xi_2E_{\mathcal{Q}}[V_t^{(1)}\int_0^t \sigma_s^{(2)}\,dW_s^{(2)}] \\
&+ \xi_1V_0^{(2)}E_{\mathcal{Q}}[\int_0^t \sigma_s^{(1)}\,dW_s^{(1)}] + c_2\xi_1V_L^{(2)}tE_{\mathcal{Q}}[\int_0^t \sigma_s^{(1)}\,dW_s^{(1)}] \\
&- \xi_1E_{\mathcal{Q}}[V_t^{(2)}\int_0^t \sigma_s^{(1)}\,dW_s^{(1)}] + \xi_1\xi_2E_{\mathcal{Q}}[\int_0^t \sigma_s^{(1)}\,dW_s^{(1)}\int_0^t \sigma_s^{(2)}\,dW_s^{(2)}]\Big]
\end{aligned}
$$

Now, we have:

$$
\begin{aligned}
E_{\mathcal{Q}}[\int_0^t \sigma_s^{(j)}\, dW_s^{(j)}] &= 0,\ j=1,2 \\
E_{\mathcal{Q}}[\int_0^t \sigma_s^{(1)}\, dW_s^{(1)} \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}] &= E_{\mathcal{Q}}\langle \int_0^t \sigma_s^{(1)}\, dW_s^{(1)} \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}\rangle = \rho_V \int_0^t ms_{12}(s)\, ds \\
E_{\mathcal{Q}}[V_t^{(1)} \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}] &= E_{\mathcal{Q}}[(V_0^{(1)} e^{-c_1 t} + V_L^{(1)}(1-e^{-c_1 t}) \\
&+ \xi_1 e^{-c_1 t} \int_0^t e^{c_1 s}\sigma_s^{(1)}\, dW_s^{(1)}) \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}] \\
&= (V_0^{(1)} e^{-c_1 t} + V_L^{(1)}(1-e^{-c_1 t}))E_{\mathcal{Q}}[\int_0^t \sigma_s^{(2)}\, dW_s^{(2)}] \\
&+ \xi_1 e^{-c_1 t} E_{\mathcal{Q}}[\int_0^t e^{c_1 s}\sigma_s^{(1)}\, dW_t^{(1)} \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}] \\
&= \xi_1 e^{-c_1 t} E_{\mathcal{Q}}\langle \int_0^t e^{c_1 s}\sigma_s^{(1)}\, dW_s^{(1)}, \int_0^t \sigma_s^{(2)}\, dW_s^{(2)}\rangle \\
&= \xi_1 \rho_V e^{-c_1 t} \int_0^t e^{c_1 s} ms_{12}(s)\, ds
\end{aligned}
$$

Similarly:

$$
E_{\mathcal{Q}}[V_t^{(2)} \int_0^t \sigma_s^{(1)}\, dW_s^{(1)}] = \xi_2 \rho_V e^{-c_2 t} \int_0^t e^{c_2 s} ms_{12}(s)\, ds
$$

Therefore:

$$
\begin{aligned}
mv_{12}^{+}(t) &:= \frac{1}{c_1 c_2}\Big[P_2(t) - (V_0^{(1)} + c_1 V_L^{(1)} t) mv_{1,2}(t) - (V_0^{(2)} + c_2 V_L^{(2)} t) mv_{1,1}(t) \\
&+ ms_{12}(t) - \xi_1\xi_2\rho_V e^{-c_1 t} B_1(t) - \xi_1\xi_2\rho_V e^{-c_2 t} B_2(t) + \xi_1\xi_2\rho_V A(t)\Big]
\end{aligned}
$$

Moreover, from equation (22):

$$
\begin{aligned}
A(t) &:= \int_0^t ms_{12}(s)\, ds = \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(t - \int_0^t e^{-\frac{1}{2}(c_1+c_2)s}\, ds\right) + \sigma_0^{(1)}\sigma_0^{(2)} \int_0^t e^{-\frac{1}{2}(c_1+c_2)s}\, ds \\
&= \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(t - \frac{2}{c_1+c_2}(1-e^{-\frac{1}{2}(c_1+c_2)t})\right) + \frac{2\sigma_0^{(1)}\sigma_0^{(2)}}{c_1+c_2}\left(1-e^{-\frac{1}{2}(c_1+c_2)t}\right) \\
B_j(t) &= \int_0^t e^{c_j s} ms_{12}(s)\, ds = \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left(\frac{1}{c_j}(e^{c_j t}-1) - \int_0^t e^{c_j - \frac{1}{2}(c_1+c_2)s}\, ds\right) \\
&+ \sigma_0^{(1)}\sigma_0^{(2)} \int_0^t e^{c_j - \frac{1}{2}(c_1+c_2)s}\, ds \\
&= \frac{\xi_1\xi_2\rho_V}{2(c_1+c_2)}\left[\frac{1}{c_j}(e^{c_j t}-1) - \frac{2(-1)^j}{c_2-c_1}(e^{\frac{1}{2}(-1)^j(c_2-c_1)t}-1)\right] \\
&+ \sigma_0^{(1)}\sigma_0^{(2)}\frac{2(-1)^j}{c_2-c_1}\left(e^{\frac{1}{2}(-1)^j(c_2-c_1)t}-1\right)
\end{aligned}
$$

for $c_1 \neq c_2$ and

$$
B_j(t) = \frac{\xi_1\xi_2\rho_V}{4c_1}\xi_1(\frac{1}{c_1}(e^{c_1 t}-1)-1) + \sigma_0^{(1)}\sigma_0^{(2)} T,\ c_1 = c_2
$$

for $c_1 = c_2$. $\square$

6.2. **Appendix B: Derivatives of the Margrabe price.** Derivatives of the Margrabe price are computed by elementary differentiation. Indeed, for the function:

$$M_4(x) = x_1 + x_2 - 2\sqrt{x_1 x_2} x_3$$

We see that:

$$\begin{aligned}
\frac{\partial M_4(x)}{\partial x_1} &= 1 - \frac{\sqrt{x_2}\, x_3}{\sqrt{x_1}}, \; \frac{\partial M_4(x)}{\partial x_2} = 1 - \frac{\sqrt{x_1}\, x_3}{\sqrt{x_2}} \\
\frac{\partial M_4(x)}{\partial x_3} &= -2\sqrt{x_1 x_2}
\end{aligned}$$

The second derivatives of $M_4(x)$ are:

$$\begin{aligned}
\frac{\partial^2 M_4(x)}{\partial x_1^2} &= \frac{\sqrt{x_2}\, x_3}{2\, {x_1}^{3/2}}, \; \frac{\partial^2 M_4(x)}{\partial x_1 \partial x_2} = 1 - \frac{x_3}{2\sqrt{x_1 x_2}} \\
\frac{\partial^2 M_4(x)}{\partial x_1 \partial x_3} &= -\frac{\sqrt{x_2}}{\sqrt{x_1}}, \; \frac{\partial^2 M_4(x)}{\partial^2 x_2} = \frac{\sqrt{x_1}\, x_3}{2\, {x_2}^{3/2}} \\
\frac{\partial^2 M_4(x)}{\partial x_2 \partial x_3} &= -\frac{\sqrt{x_1}}{\sqrt{x_2}}, \; \frac{\partial^2 M_4(x)}{\partial x_3^2} = 0
\end{aligned}$$

We denote $M_3 = \log\left(\frac{S_0^{(1)}}{S_0^{(2)}}\right)$ and introduce the function:

$$d_1(x) \;=\; M_3 M_4^{-\frac{1}{2}}(x) + \frac{1}{2} M_4^{\frac{1}{2}}(x)$$

The first and second derivatives of $d_1(x)$ are:

$$\begin{aligned}
\frac{\partial d_1(x)}{\partial x_j} &= -\frac{1}{2} M_3 M_4^{-\frac{3}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} + \frac{1}{4} M_4^{-\frac{1}{2}}(x) \frac{\partial M_4(x)}{\partial x_j}, j = 1,2,3 \\
\frac{\partial^2 d_1(x)}{\partial x_j \partial x_k} &= \frac{3}{4} M_3 M_4^{-\frac{5}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} \frac{\partial M_4(x)}{\partial x_k} - \frac{1}{2} M_3 M_4^{-\frac{3}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k} \\
&- \frac{1}{8} M_4^{-\frac{3}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} \frac{\partial M_4(x)}{\partial x_k} + \frac{1}{4} M_4^{-\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k}, j,k = 1,2,3
\end{aligned}$$

Moreover:

$$\begin{aligned}
d_2(x) &= d_1(x) - M_4^{\frac{1}{2}}(x) \\
\frac{\partial d_2(x)}{\partial x_j} &= \frac{\partial d_1(x)}{\partial x_j} - \frac{1}{2} M_4^{-\frac{1}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} \\
\frac{\partial^2 d_2(x)}{\partial x_j \partial x_k} &= \frac{\partial^2 d_1(x)}{\partial x_j \partial x_k} + \frac{1}{4} M_4^{-\frac{3}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} \frac{\partial M_4(x)}{\partial x_k} \\
&- \frac{1}{2} M_4^{-\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k}, j,k = 1,2,3
\end{aligned}$$

Finally:

$$
\begin{aligned}
\frac{\partial C_M(x)}{\partial x_j} &= M_1 f_Z(d_1(x))\frac{\partial d_1(x)}{\partial x_j} - M_2 f_Z(d_2(x))\frac{\partial d_2(x)}{\partial x_j}, j = 1,2,3 \\
\frac{\partial^2 C_M(x)}{\partial x_j \partial x_k} &= M_1\left(\frac{\partial f_Z(d_1(x))}{\partial x_k}\frac{\partial d_1(x)}{\partial x_j} + f_Z(d_1(x))\frac{\partial^2 d_1(x)}{\partial x_j \partial x_k}\right) \\
&- M_2\left(\frac{\partial f_Z(d_2(x))}{\partial x_k}\frac{\partial d_2(x)}{\partial x_j} + f_Z(d_2(x))\frac{\partial^2 d_2(x)}{\partial x_j \partial x_k}\right) \\
&= M_1\left(-d_1(x) f_Z(d_1(x))\frac{\partial d_1(x)}{\partial x_j}\frac{\partial d_1(x)}{\partial x_k} + f_Z(d_1(x))\frac{\partial^2 d_1(x)}{\partial x_j \partial x_k}\right) \\
&- M_2\left(-d_2(x) f_Z(d_2(x))\frac{\partial d_2(x)}{\partial x_j}\frac{\partial d_2(x)}{\partial x_k} + f_Z(d_2(x))\frac{\partial^2 d_2(x)}{\partial x_j \partial x_k}\right)
\end{aligned}
$$

where f_Z is the probability density function of a standard normal distribution and $M_j = e^{-rT} S_0^{(j)},\ j = 1,2.$

References

[1] Alvarez A., Escobar M., and Olivares P. Pricing two dimensional derivatives under stochastic correlation. *International Journal of Financial Markets and Derivatives*, 2(4):265–287, 2011.

[2] C. Bernard and Z. Cui. A note on exchange options under stochastic interest rates (june 16, 2010). Technical report, Available at SSRN, 06 2010.

[3] R. Caldana, G.H.L. Cheang, C. Chiarella, and G. Fusai. Correction: Exchange option under jump-diffusion dynamics. *Applied Mathematical Finance*, 22(1):99–103, 2015.

[4] G.H.L. Cheang and C.Chiarella. Exchange options under jump-diffusion dynamics. *Applied Mathematical Finance*, 18(3):245–276, 2011.

[5] Kim G. Valuation of exchange option with credit risk in a hybrid model. *Mathematics*, 8(2091), 2020.

[6] S. L. Heston. A closed-form solution for options with stochastic volatility withapplications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[7] J.C. Hull and A. White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987.

[8] J. Kim and Chang-Rae Park. A multiscale extension of the margrabe formula under stochastic volatility. *Chaos, Solitons and Fractals: Nonlinear Science, and Nonequilibrium and Complex Phenomena*, 97:59–65, 2017.

[9] Deng S Li M and Zhou J. Closed-form approximations for spread options prices and greeks. *Journal of Derivatives*, 15(3):58–80, 2008.

[10] W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33(1):177–186, 1978.

[11] Ju N. Pricing asian and basket options via taylor expansion. *Journal of Computational Finance*, 5:79–103, 2002.

[12] P. Olivares. Pricing of basket options under a jump-diffusion model. *Book: Polynomials-Theory and Applications. IntechOpen*, 2018.

[13] P. Olivares and A. Alvarez. Pricing basket options by polynomial approximations. *Journal of Applied Mathematics*, 2016. Article ID 9747394.

[14] P Olivares and C. Diaz. A finite element approach for spread contract valuation via associated two-dimensional pide. *Comp. Appl. Math.*, 42(15), 2023.

[15] Pasricha P. and A. Goel. Pricing vulnerable power exchange options in an intensity based framework. *Journal of Computational and Applied Mathematics*, 355:106–15, 2019.

[16] Taschini Palmer and Urech. Conservation payments under uncertainty by engel. *Land Economics*, 91(1):36–56, 2015.

[17] P. Pasricha and A. Goel. A closed-form pricing formula for european exchange options with stochastic volatility. *Probability in the Engineering and Informational Sciences*, 36:606–15, 2022.

[18] Fazlollah Soleymani and Andrey Itkin. Pricing foreign exchange options under stochastic volatility and interest rates using an rbf–fd method. *Journal of Computational Science*, 37:101028, 2019.

[19] E Villamor and P. Olivares. Valuing exchange options under an ornstein-uhlenbeck covariance model. *Int. J. Financial Stud.*, 11(2), 2023.

[20] Guanying Wang, Xingchun Wang, and Xinjian Shao. Exchange options for catastrophe risk management. *The North American Journal of Economics and Finance*, 59(C):S1062940821001832, 2022.